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Fixed Point Results in Soft Metric Space

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Abstract

In the present paper some theorems on soft metric spaces are established for Non-expansive mapping. The obtained results are generalized form of some basic fundamental results in fixed point theory and very useful in engineering and agriculture studies. The results are established at the basis of soft set principle.

Keywords: Non-expansive mapping; soft metric space, fixed point.

1. Introduction and Preliminaries:

The generalization of fixed point theorems for non- expansive mappings are very useful not only in functional analysis but also in engengering, management, pure and applied physics.
Suzuki [15] defined and explained A class of generalized non expansive mappings on a nonempty subset C of a Banach space X in 2008. The detail about soft sets, soft fixed point, soft metric space can be seen in [3,4,22. 17]

Definition 1.1: (See [14]) Let M be a nonempty subset of a Soft Metric space$(\tilde{X}, \tilde{d}, E)$. A mapping $F: M \times M \to \tilde{X}$ is called a non expansive if$\tilde{d}\left(Fx_{s_1}, Fy_{s_2}\right) \leq \tilde{d}(x_{s_1}, y_{s_2})$, for all $x_{s_1}, y_{s_2} \in M$.

Definition 1.2: (See [14]) For a nonempty subset M of a soft metric space$(\tilde{X}, \tilde{d}, E)$, a mapping $F: M \times M \to \tilde{X}$is said to be quasi non expansive if$\tilde{d}(Fx_{s_1}, z_{s_3}) \leq \tilde{d}(x_{s_1}, z_{s_3})$, for all $x_{s_1} \in M$ and $z_{s_3} \in SF(E)$, (where SF(E)denotes the set of all Soft points of E).

Definition 1.3:
(See [14]) For a nonempty subset M of a soft Metric space$(\tilde{X}, \tilde{d}, E)$, a mapping $F: M \to \tilde{X}$ is said to satisfy the contractive-condition (CC) on M if $\frac{1}{2}\tilde{d}(x_{s_1}, Fx_{s_1}) \leq d(x_{s_1}, y_{s_2})$ implies$\tilde{d}(Fx_{s_1}, Fy_{s_2}) \leq d(x_{s_1}, y_{s_2})$ for all$x_{s_1}, y_{s_2} \in M$.

Obviously every non expansive mapping satisfies the contractive condition (CC) on M. But there are also some non continuous mappings satisfying the condition (CC) (see [16]).

***Definition 1.4*:**

(See [7]) For a nonempty subset M of a soft metric space$(\tilde{X}, \tilde{d}, E)$ and α ∈ (0, 1), a mapping $F: M \to \tilde{X}$ is said to satisfy (CC_λ)-contractive condition on M if $\alpha\tilde{d}(x_{s_1}, Fx_{s_1}) \leq d(x_{s_1}, y_{s_2})$ implies$\tilde{d}(Fx_{s_1}, Fy_{s_2}) \leq d(x_{s_1}, y_{s_2})$, for all $x_{s_1}, y_{s_2} \in M$

Definition 1.5:

(See [7]) If M is a closed convex and bounded subset of $(\tilde{X}, \tilde{d}, E)$, and a self mapping F on M is non expansive, then there exists a sequence $\{\tilde{x}_{\lambda_n}^n\}$ in M such that$\tilde{d}(\tilde{x}_{\lambda_n}^n - F\tilde{x}_{\lambda_n}^n) \to 0$, such a sequence is called almost soft point sequence for F.

Definition 1.6:

(See [7]) Let M be a nonempty subset of a soft Metric space $(\tilde{X}, \tilde{d}, E)$ and $\{\tilde{x}_{\lambda_n}^n\}$ be a bounded sequence in $(\tilde{X}, \tilde{d}, E)$. For each$x_\lambda \in \tilde{X}$, we have

(i) asymptotic radius of $\{\tilde{x}_{\lambda_n}^n\}$ at x_λ is defined by $r(x_\lambda, \{\tilde{x}_{\lambda_n}^n\}) = \lim_{n\to\infty} \sup d(\tilde{x}_{\lambda_n}^n, x_\lambda)$

(ii) asymptotic radius of $\{\tilde{x}_{\lambda_n}^n\}$ relative to M is defined by $r(M, \{\tilde{x}_{\lambda_n}^n\}) = \lim_{n\to\infty} \inf$ r $(x_\lambda, \tilde{x}_{\lambda_n}^n): x_\lambda$ ε M .

(iii) asymptotic center of $\{\tilde{x}_{\lambda_n}^n\}$relative to M is defined by $A(M, \{\tilde{x}_{\lambda_n}^n\}) = \{x_\lambda \in M: r(x_\lambda, \{\tilde{x}_{\lambda_n}^n\}) = r(M, \{\tilde{x}_{\lambda_n}^n\})\}$.

We note that $A(M, \{\tilde{x}_{\lambda_n}^n\})$ is nonempty. Again, if $\tilde{X}$ is uniformly convex, then $A(M, \{\tilde{x}_{\lambda_n}^n\})$has exactly one point.

Definition 1.7:

(See [14]) A Soft Metric space $(\tilde{X}, \tilde{d}, E)$ is said to satisfy the opial property if, for every sequence $\{\tilde{x}_{\lambda_n}^n\}$ in $\tilde{X}$with $\tilde{x}_{\lambda_n}^n \to z_\mu$, we have $\lim_{n\to\infty} \widetilde{\inf d}\, (\tilde{x}_{\lambda_n}^n, z_\mu) \leq \lim_{n\to\infty} \widetilde{\inf d}\, (\tilde{x}_{\lambda_n}^n, y_\eta)$ whenever $y_\eta \neq z_\mu$.

2. Some modified Results:

2.1. Definition:

Let M be a nonempty subset of a soft metric space$(\tilde{X}, \tilde{d}, E)$. Let γ ∈ [0, 1] & μ ∈ $\left[0, \frac{1}{2}\right]$ be soft real numbers such that 2μ ≤ γ, a soft mapping $F: M \to \tilde{X}$is said to satisfy the condition$B_{\gamma,\mu}$on M if, for all $\tilde{x}_\lambda$, y_η in M,

$$\gamma\tilde{d}(\tilde{x}_\lambda, F\tilde{x}_\lambda) \leq \tilde{d}(\tilde{x}_\lambda, y_\eta) + \mu\tilde{d}(y_\eta, Fy_\eta), \text{ implies } \tilde{d}(F\tilde{x}_\lambda, F\tilde{y}_\eta) \leq (1-\gamma)\tilde{d}(\tilde{x}_\lambda, \tilde{x}_\lambda) + \mu\left(\tilde{d}(\tilde{x}_\lambda, F\tilde{x}_\lambda) + \tilde{d}(\tilde{x}_\lambda, F\tilde{x}_\lambda)\right)$$

Clearly, this class includes the class of non-expansive mappings (for γ = μ = 0).
Also, if a mapping satisfies the contractive condition (CC), then it will satisfy the condition $\boldsymbol{B_{\gamma,\mu}}$ for γ = μ = 0.
As $\frac{1}{2}\tilde{d}(\tilde{x}_\lambda, F\tilde{x}_\lambda) \leq \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\eta) \Rightarrow \tilde{d}(F\tilde{x}_\lambda, F\tilde{y}_\eta) \leq \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\eta)$ for (CC) condition
So, clearly, $\tilde{d}(F\tilde{x}_\lambda, F\tilde{y}_\eta) \leq (1-\gamma)\tilde{d}(\tilde{x}_\lambda, \tilde{y}_\eta) + \mu\left(\tilde{d}(\tilde{x}_\lambda, F\tilde{y}_\eta) + \tilde{d}(\tilde{y}_\eta, F\tilde{x}_\lambda)\right)$ for γ = μ = 0, but the converse is not true.

2.2. Lemma: Let M≠∅ ⊂ $\tilde{X}$, *F: M→$\tilde{X}$ which satisfy $B_{\gamma,\mu}$condition. If z_ρ is a soft point of F on M, then,*

$\tilde{d}(z_\rho, Fx_\lambda) \leq \tilde{d}(z_\rho, x_\lambda)$, *for all $x_\lambda \in M$.*

Proof: Since $\gamma\tilde{d}(z_\rho, Fz_\rho) = 0 \leq \tilde{d}(z_\rho, x_\lambda) + \mu\tilde{d}(x_\lambda, Fx_\lambda)$. By $B_{\gamma,\mu}$condition,

$$\begin{aligned}\tilde{d}(Fx_\lambda, Fy_\eta) &\leq (1-\gamma)\tilde{d}(z_\rho, x_\lambda) + \mu\left(\tilde{d}(x_\lambda, Fz_\rho) + \tilde{d}(z_\rho, Fx_\lambda)\right)\\ &= (1-\gamma)\tilde{d}(z_\rho, x_\lambda) + \mu\left(\tilde{d}(x_\lambda, z_\rho) + \tilde{d}(z_\rho, Fx_\lambda)\right)\end{aligned} \Rightarrow \tilde{d}(z_\rho, Fx_\lambda) \leq$$

$$\left(\frac{1-\gamma+\mu}{1-\mu}\right)\tilde{d}(z_\rho, x_\lambda) \leq \tilde{d}(z_\rho, x_\lambda) \text{ (as } 2\mu \leq \gamma)$$

So F is quasi-non expansive. However, the converse of above *Lemma* does not hold in general

Example

Let F: [0, 4]⇢[0,4] defined by $Fx_\lambda = \begin{cases} 0 & \text{if } x_\lambda \neq 4 \\ 3 & \text{if } x_\lambda = 4 \end{cases}$. Then F has a soft fixed point at $x_\lambda = 0$, and also $F(x_\lambda) \leq x_\lambda$, $\forall x_\lambda \in [0,4]$.

Hence, F is quasi-non expansive. We show that F does not satisfy the condition $\boldsymbol{B_{\gamma,\mu}}$.

For $x_\lambda = 4, y_\eta = 3$

$$\gamma\tilde{d}(x_\lambda, Fx_\lambda) = \gamma \leq 1 + 3\mu = \tilde{d}(4,3) + \mu\tilde{d}(3, F(3))$$

But $\tilde{d}(Fx_\lambda, Fy_\eta) \leq \tilde{d}(F(4), F(3)) = 3$ and

$$(1-\gamma)\tilde{d}(x_\lambda, y_\eta) + \mu\left(\tilde{d}(x_\lambda, Fy_\eta) + \tilde{d}(y_\eta, Fx_\lambda)\right) = 1-\gamma+4\mu \leq 1-\gamma+2\gamma \quad (\because 2\mu \leq \gamma)$$

< 3 (as γ ε [0,1])

$$= \tilde{d}(Fx_\lambda, Fy_\eta)$$

So, $\boldsymbol{B_{\gamma,\mu}}$condition is not satisfied. The following are some basic properties of mappings which satisfy the condition $\boldsymbol{B_{\gamma,\mu}}$ on M.

2.3. Proposition:

Let M be a nonempty subset of a soft Metric space$\widetilde{\boldsymbol{X}}$. Let F: M →M satisfy the condition $\boldsymbol{B_{\gamma,\mu}}$on M, Then, for all $\boldsymbol{x_\lambda, y_\eta} \in \boldsymbol{M}$and for, some soft number m ∈ [0, 1]

(i) $\widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, F^2x_\lambda}) \leq \widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda})$

(ii) at least one of the following ((a) and (b)) holds:

(a) $\frac{\boldsymbol{m}}{\boldsymbol{2}}\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) \leq \tilde{d}(\boldsymbol{x_\lambda, y_\eta})$

(b) $\frac{\boldsymbol{m}}{\boldsymbol{2}}\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, F^2x_\lambda}) \leq \tilde{d}(\boldsymbol{Fx_\lambda, y_\eta})$

The condition (a) implies $\widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, Fy_\eta}) \leq \left(\boldsymbol{1-\frac{m}{2}}\right)\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, y_\eta}) + \boldsymbol{\mu}\left[\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fy_\eta}) + \widetilde{\boldsymbol{d}}(\boldsymbol{y_\eta, Fx_\lambda})\right]$

and

The condition (b) implies $\widetilde{\boldsymbol{d}}(\boldsymbol{F^2x_\lambda, Fy_\eta}) \leq \left(\boldsymbol{1-\frac{m}{2}}\right)\widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, y_\eta}) + \boldsymbol{\mu}\left[\widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, Fy_\eta}) + \widetilde{\boldsymbol{d}}(\boldsymbol{y_\eta, F^2x_\lambda})\right]$

iii) $\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fy_\eta}) \leq (\boldsymbol{3-m})\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) + (\boldsymbol{1-\frac{m}{2}})\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, y_\eta}) + \boldsymbol{\mu}[\boldsymbol{2}\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) + \widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fy_\eta}) + \widetilde{\boldsymbol{d}}(\boldsymbol{y_\eta, Fx_\lambda}) + \boldsymbol{2}\widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, F^2x_\lambda})]$

Proof: i) We have, for all $\boldsymbol{x_\lambda} \in \boldsymbol{M}$

$$\boldsymbol{\gamma}\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) \leq \widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) + \boldsymbol{\mu}\widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, F^2x_\lambda})$$

So by the condition $\boldsymbol{B_{\gamma,\mu}}$ (replacing$\boldsymbol{y_\eta}$ by Fx$_{\boldsymbol{\lambda}}$)

$$\tilde{d}(\mathrm{Fx}_\lambda, F^2\mathrm{x}_\lambda) \leq (1-\gamma)\tilde{d}(\mathrm{x}_\lambda, F\mathrm{x}_\lambda) + \mu\tilde{d}(\mathrm{x}_\lambda, F^2\mathrm{x}_\lambda)$$
$$\leq (1-\gamma)\tilde{d}(\mathrm{x}_\lambda, F\mathrm{x}_\lambda) + \mu\left(\tilde{d}(\mathrm{x}_\lambda, F\mathrm{x}_\lambda) + \tilde{d}(F\mathrm{x}_\lambda, F^2\mathrm{x}_\lambda)\right)$$

$$\Rightarrow \widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, F^2x_\lambda}) \leq \left(\frac{1-\boldsymbol{\gamma}+\boldsymbol{\mu}}{\boldsymbol{1-\mu}}\right)\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) \leq \widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda})$$

ii) We assume on the contrary that:

$\frac{\boldsymbol{m}}{\boldsymbol{2}}\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) > \tilde{d}(\boldsymbol{x_\lambda, y_\eta})$ and$\frac{\boldsymbol{m}}{\boldsymbol{2}}\widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, F^2x_\lambda}) > \tilde{d}(\boldsymbol{Fx_\lambda, y_\eta})$, for some $\boldsymbol{x_\lambda, y_\eta} \in \boldsymbol{M}$.

Now

$$\tilde{d}(x_\lambda, Fx_\lambda) \leq \tilde{d}(x_\lambda, y_\eta) + \tilde{d}(y_\eta, Fx_\lambda) \prec \frac{m}{2}\tilde{d}(x_\lambda, Fx_\lambda) + \frac{m}{2}\tilde{d}(Fx_\lambda, F^2x_\lambda) \qquad \text{(by (i))}$$
$$\prec \frac{m}{2}\tilde{d}(x_\lambda, Fx_\lambda) + \frac{m}{2}\tilde{d}(x_\lambda, Fx_\lambda) \leq \tilde{d}(x_\lambda, Fx_\lambda) \qquad \text{since } m \leq 1$$

That is$\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) < \widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda})$. This is not possible, so, at least one of (a) and (b) holds.

iii) $\widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fy_\eta}) \leq \widetilde{\boldsymbol{d}}(\boldsymbol{x_\lambda, Fx_\lambda}) + \widetilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda, Fy_\eta})$

If (ii) (a) holds,

$$\tilde{d}(\mathrm{x}_\lambda, \mathrm{Fx}_\lambda) \leq \tilde{d}(\mathrm{x}_\lambda, \mathrm{Fx}_\lambda) + \left(1-\frac{\mathrm{m}}{2}\right)\tilde{d}(\mathrm{x}_\lambda, \mathrm{y}_\eta) + \mu\left[\tilde{d}(\mathrm{x}_\lambda, \mathrm{Fy}_\eta) + \tilde{d}(\mathrm{y}_\eta, \mathrm{Fx}_\lambda)\right]$$
$$\leq (3-m)\tilde{d}(x_\lambda, Fx_\lambda) + \left(1-\frac{m}{2}\right)\tilde{d}(x_\lambda, y_\eta) + \mu\left[2\tilde{d}(x_\lambda, Fy_\eta) + \tilde{d}(x_\lambda, Fy_\eta) + \tilde{d}(y_\eta, Fx_\lambda) + 2\tilde{d}(Fx_\lambda, F^2x_\lambda)\right]$$

If (ii) (b) holds,

$$\tilde{d}(x_\lambda, Fy_\eta) \le \tilde{d}(x_\lambda, Fx_\lambda) + \tilde{d}(Fx_\lambda, F^2x_\lambda) + \tilde{d}(F^2x_\lambda, Fy_\eta)$$

$$\le \tilde{d}(x_\lambda, Fx_\lambda) + \left(1-\frac{m}{2}\right)\tilde{d}(Fx_\lambda, x_\lambda) + \mu\left[\tilde{d}(Fx_\lambda, Fx_\lambda) + \tilde{d}(x_\lambda, F^2x_\lambda)\right]$$

$$\le (3-m)\tilde{d}(x_\lambda, Fx_\lambda) + \left(1-\frac{m}{2}\right)\tilde{d}(x_\lambda, y_\eta) + \mu\left[2\tilde{d}(x_\lambda, Fx_\lambda) + \tilde{d}(x_\lambda, Fy_\eta) + \tilde{d}(y_\eta, Ex_\lambda) + 2\tilde{d}(Fx_\lambda, F^2x_\lambda)\right]$$

$$+\left(1-\frac{m}{2}\right)\tilde{d}(Fx_\lambda, y_\eta) + \mu\left[\tilde{d}(Fx_\lambda, Fy_\eta) + \tilde{d}(y_\eta, F^2x_\lambda)\right]$$

$$= (\mathbf{3}-\boldsymbol{c})\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Fx_\lambda}) + \left(\mathbf{1}-\frac{\boldsymbol{m}}{\mathbf{2}}\right)\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{y_\eta}) + \boldsymbol{\mu}[\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{F^2x_\lambda}) + \tilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda}, \boldsymbol{Fy_\eta}) + \tilde{\boldsymbol{d}}(\boldsymbol{y_\eta}, \boldsymbol{F^2x_\lambda})]$$

$$\le (\mathbf{3}-\boldsymbol{c})\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Ex_\lambda}) + \left(\mathbf{1}-\frac{\boldsymbol{m}}{\mathbf{2}}\right)\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{y_\eta})$$

$$+\boldsymbol{\mu}[\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Fx_\lambda}) + \tilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda}, \boldsymbol{F^2x_\lambda}) + \tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{F^2x_\lambda}) + \tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Fx_\lambda}) + \tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Fy_\eta}) + \tilde{\boldsymbol{d}}(\boldsymbol{y_\eta}, \boldsymbol{Fx_\lambda}) + \tilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda}, \boldsymbol{F^2x_\lambda})]$$

$$= (\mathbf{3}-\boldsymbol{c})\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Fx_\lambda}) + \left(\mathbf{1}-\frac{\boldsymbol{m}}{\mathbf{2}}\right)\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{y_\eta}) + \boldsymbol{\mu}[\mathbf{2}\tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Fx_\lambda}) + \tilde{\boldsymbol{d}}(\boldsymbol{x_\lambda}, \boldsymbol{Fy_\eta}) + \tilde{\boldsymbol{d}}(\boldsymbol{y_\eta}, \boldsymbol{Fx_\lambda}) + \mathbf{2}\tilde{\boldsymbol{d}}(\boldsymbol{Fx_\lambda}, \boldsymbol{F^2x_\lambda})]$$

2.4. Proposition:

Let M be a nonempty convex and bounded subset of a soft metric space $\tilde{X}$ and F be a self-mapping on M. We assume that F satisfies the condition Bγ,μ on M. For $x_{\lambda_0}^0 \in M$, let a sequence $\{x_{\lambda_n}^n\}$ in M be defined by

$$x_{\lambda_{n+1}}^{n+1} = \rho Fx_{\lambda_n}^n + (1-\rho)x_{\lambda_n}^n, \qquad (2.5.1)$$

Where$\rho \in \gamma, 1) - \{0\}, n \in N \cup \{0\}$, then $\tilde{d}(Fx_{\lambda_n}^n, x_{\lambda_n}^n) \to 0$as $n \to \infty$

Proof: Since $\rho \ge \gamma$we have

$$\gamma\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) \le \rho\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) = \tilde{d}(x_{\lambda_n}^n, x_{\lambda_{n+1}}^{n+1}) \qquad \text{(By (2.5.1))}$$

i.e. $$\gamma\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) \le d(x_{\lambda_n}^n, x_{\lambda_{n+1}}^{n+1}) + \mu d(x_{\lambda_{n+1}}^{n+1}, Fx_{\lambda_{n+1}}^{n+1})$$

So by the condition $B_{\gamma,\mu}$ (for$y_{\eta_n}^n = x_{\lambda_{n+1}}^{n+1}$)

$$\tilde{d}(Fx_{\lambda_{n+1}}^{n+1}, Fx_{\lambda_{n+1}}^{n+1}) \le (1-\gamma)\tilde{d}(x_{\lambda_n}^n, x_{\lambda_{n+1}}^{n+1}) + \mu[\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_{n+1}}^{n+1}) + \tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n)]$$

$$\tilde{d}\left(Fx_{\lambda_n}^n, \frac{1}{\rho}(x_{\lambda_{n+2}}^{n+2} - (1-\rho)x_{\lambda_{n+1}}^{n+1})\right) \le \tilde{d}(x_{\lambda_n}^n, x_{\lambda_{n+1}}^{n+1}) + \mu\left[\tilde{d}\left(x_{\lambda_n}^n, \frac{1}{\rho}(x_{\lambda_{n+2}}^{n+2} - (1-\rho)x_{\lambda_{n+1}}^{n+1})\right) + d(x_{\lambda_{n+1}}^{n+1}, Fx_{\lambda_n}^n)\right]$$

For which we get $(1-\mu)\lim_{n\to\infty}\tilde{d}(Fx_{\lambda_n}^n, x_{\lambda_n}^n) \le 0$

⇨ $\lim_{n\to\infty}\tilde{d}(Fx_{\lambda_n}^n, x_{\lambda_n}^n) = 0$ (Since$\mu \ne 1$)

2.5. Theorem

Let M be a compact and convex subset of a soft Metric space$\tilde{\boldsymbol{X}}$. Let F be a self-mapping on M satisfying the condition$B\gamma, \mu$. For $x_{\lambda_0}^0 \in M$, let$\{x_{\lambda_n}^n\}$be a sequence in M as defined in Proposition 2.6, where γ is sufficiently small. Then$\{x_{\lambda_n}^n\}$converges strongly to a fixed point of F.

Proof: Since M is compact, there exists a subsequence $\left\{x_{\lambda_{n_j}}^{n_j}\right\}$of $\{x_{\lambda_n}^n\}$and $z_\rho \in M$ such that $\left\{x_{\lambda_{n_j}}^{n_j}\right\}$converges to z_ρ (see 15).

Now, by proposition for$\gamma = \frac{m}{2}$,m ε [0,1]

$$\gamma\tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fx^{n_j}_{\lambda_{n_j}}) \lesseqqgtr \tilde{d}(x^{n_j}_{\lambda_{n_j}}, z_\rho) \text{ Implies } \gamma\tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fx^{n_j}_{\lambda_{n_j}}) \leq \tilde{d}(x^{n_j}_{\lambda_{n_j}}, z_\rho) + \mu\tilde{d}(z_\rho, Fz_\rho)$$

So by the condition $B\gamma, \mu$ we have $\tilde{d}(Fx^{n_j}_{\lambda_{n_j}}, Fz_\rho) \leq (1-\gamma)\tilde{d}(x^{n_j}_{\lambda_{n_j}}, z_\rho) + \mu\left[\tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fz_\rho) + \tilde{d}(z_\rho, Fx^{n_j}_{\lambda_{n_j}})\right]$

Again

$$\tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fz_\rho) \leq \tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fx^{n_j}_{\lambda_{n_j}}) + \tilde{d}(Fx^{n_j}_{\lambda_{n_j}}, Fz_\rho)$$

$$\leq \tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fx^{n_j}_{\lambda_{n_j}}) + (1-\gamma)\tilde{d}(x^{n_j}_{\lambda_{n_j}}, z_\rho) + \mu\left[\tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fz_\rho) + \tilde{d}(z_\rho, Fx^{n_j}_{\lambda_{n_j}})\right]$$

$$\leq \tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fx^{n_j}_{\lambda_{n_j}}) + (1-\gamma)\tilde{d}(x^{n_j}_{\lambda_{n_j}}, z_\rho) + \mu\left[\tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fz_\rho) + \tilde{d}(z_\rho, x^{n_j}_{\lambda_{n_j}}) + \tilde{d}(x^{n_j}_{\lambda_{n_j}}, Fx^{n_j}_{\lambda_{n_j}})\right]$$

So, taking $n_j \to \infty$and using proposition, we get

$$(1-\mu)\tilde{d}(z_\rho, Fz_\rho) \leq 0 \text{Implies that } Fz_\rho = z_\rho \text{ (Since} \mu \neq 1)$$

Showing that z_ρ is soft point for F. Now

$$\tilde{d}(x^{n+1}_{\lambda_{n+1}}, z_\rho) \leq \rho\tilde{d}(Fx^n_{\lambda_n}, z_\rho) + (1-\rho)\tilde{d}(x^n_{\lambda_n}, z_\rho)$$
$$\leq \rho\tilde{d}(x^n_{\lambda_n}, z_\rho) + (1-\rho)\tilde{d}(x^n_{\lambda_n}, z_\rho)$$
$$= \tilde{d}(x^n_{\lambda_n}, z_\rho) \qquad \text{for all n } \varepsilon \text{ N} \cup \{0\}$$

Thus is a monotonically decreasing sequence of nonnegative real numbers and will converge to some real, say u. Now

$$\tilde{d}(x^n_{\lambda_n}, Fz_\rho) \leq \tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n}) + (1-\gamma)\tilde{d}(x^n_{\lambda_n}, z_\rho) + \mu\left[\tilde{d}(x^n_{\lambda_n}, Fz_\rho) + \tilde{d}(z_\rho, x^n_{\lambda_n}) + \tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n})\right]$$

$$\Rightarrow \tilde{d}(x^n_{\lambda_n}, z_\rho) \leq \tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n}) + (1-\gamma)\tilde{d}(x^n_{\lambda_n}, z_\rho) + \mu\left[2\tilde{d}(x^n_{\lambda_n}, z_\rho) + \tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n})\right]$$

Taking limit as n → ∞, we get $u \leq (1-\gamma)u + \mu(2u) \Rightarrow (\gamma - 2\mu) \leq 0$, which is possible only for $u = 0$, since $2\mu \leq \gamma$. Hence, $\{x^n_{\lambda_n}\}$converges strongly to z_ρ.

2.6. Theorem

Let M be a weakly compact and convex subset of a uniformly convex soft Metric Space$\tilde{X}$. Let F be a self-mapping on M satisfying the condition Bγ ,μ. Then F has a soft fixed point.

Proof: Consider the sequence $\{x^n_{\lambda_n}\}$ in M as defined in Proposition

Then, $\widetilde{\lim\sup}\, d\,(Fx^n_{\lambda_n}, x^n_{\lambda_n}) = 0$

As in [7], let g be a continuous convex function from M into [0, ∞) defined by

$$g(x^n_{\lambda_n}) = \lim_{n\to\infty}\sup d\,(x^n_{\lambda_n}, x_\lambda), \text{ for all } x_\lambda \in M$$

Again, since M is weakly compact and g is weakly lower semi-continuous, there is $z_\rho \in M$such that:

$$g(z_\rho) = \min\{g(x_\lambda) : x_\lambda \in C\}$$

$$\tilde{d}(x^n_{\lambda_n}, Fz_\rho) \leq (3-2\gamma)\tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n}) + (1-\gamma)\tilde{d}(x^n_{\lambda_n}, z_\rho) + \mu\left[2\tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n}) + \tilde{d}(x^n_{\lambda_n}, z_\rho) + \tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n}) + 2\tilde{d}(x^n_{\lambda_n}, Fx^n_{\lambda_n})\right]$$

So, $(1-\mu)\lim_{n\to\infty}\widetilde{\sup d}\,(x^n_{\lambda_n}, Fz_\rho) \leq (1-\gamma+\mu)\lim_{n\to\infty}\widetilde{\sup d}\,(x^n_{\lambda_n}, z_\rho)$

$$\Rightarrow \lim_{n\to\infty}\widetilde{\sup d}\,(x^n_{\lambda_n}, Fz_\rho) \leq \frac{(1-\gamma+\mu)}{1-\mu}\lim_{n\to\infty}\widetilde{\sup d}\,(x^n_{\lambda_n}, z_\rho)$$

$$\Rightarrow g(Fz_\rho) \leq g(z_\rho)$$

Since $g(z_\rho)$ is the minimum. $g(Fz_\rho) = g(z_\rho)$
Now, if $Fz_\rho \neq z_\rho$, then as g is strictly quasi-convex, we have

$$g(z_\rho) \leq g(\rho F z_\rho + (1 - \rho)z_\rho) < max\{g(z_\rho), g(Fz_\rho)\} = g(z_\rho)$$

Which is a contradiction, hence $Fz_\rho = z_\rho$.

2.7. Theorem

Let M be a nonempty subset of a soft Metric space $\tilde{X}$ having the Opial property. Let F be a self-mapping on M satisfying the condition Bγ,μ. If $\{x_{\lambda_n}^n\}$ is a sequence in $\tilde{X}$ such that:

(i) $\{\boldsymbol{x_{\lambda_n}^n}\}$ Converges weakly to$\boldsymbol{z_\rho}$,
(ii) $\boldsymbol{\lim_{n\to\infty} \tilde{d}(Fx_{\lambda_n}^n, \{X_{\lambda_n}^n\}) = 0}$,Then $\boldsymbol{Fz_\rho = z_\rho}$.

Proof:

$$\boldsymbol{\gamma\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) \leq \tilde{d}(x_{\lambda_n}^n, z_\rho) \leq \tilde{d}(x_{\lambda_n}^n, z_\rho) + \mu\tilde{d}(z_\rho, Fz_\rho)}.$$

So, by the condition Bγ,μ

$$\boldsymbol{\tilde{d}(Fx_{\lambda_n}^n, Fz_\rho) \leq (1-\gamma)\tilde{d}(x_{\lambda_n}^n, z_\rho) + \mu\left[\tilde{d}(x_{\lambda_n}^n, Fz_\rho) + \tilde{d}(z_\rho, Fx_{\lambda_n}^n)\right]} \quad (2.8.1)$$

Now

$$\tilde{d}(x_{\lambda_n}^n, Fz_\rho) \leq \tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) + \tilde{d}(Fx_{\lambda_n}^n, Fz_\rho)$$

$$\leq \tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) + (1-\gamma)\tilde{d}(x_{\lambda_n}^n, z_\rho) + \mu\left[\tilde{d}(x_{\lambda_n}^n, Fz_\rho) + \tilde{d}(z_\rho, x_{\lambda_n}^n) + \tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n)\right]$$

by 2.8.1

So, taking limit as n→∞ and using (ii), we get

$$\boldsymbol{\tilde{d}(x_{\lambda_n}^n, Fz_\rho) \leq \frac{1-\gamma+\mu}{1-\mu}\tilde{d}(x_{\lambda_n}^n, z_\rho) \leq \tilde{d}(x_{\lambda_n}^n, z_\rho)}$$

So, $$\boldsymbol{\lim_{n\to\infty} \widetilde{\inf d}\,(x_{\lambda_n}^n, Fz_\rho) \leq \lim_{n\to\infty} \widetilde{\inf d}\,(x_{\lambda_n}^n, z_\rho)} \quad (2.8.2)$$

Let $\boldsymbol{Fz_\rho \neq z_\rho}$ Since $\boldsymbol{x_{\lambda_n}^n \to z_\rho}$ (weakly) the opial property, we have

$$\boldsymbol{\lim_{n\to\infty} \widetilde{\inf d}\,(x_{\lambda_n}^n, z_\rho) \leq \lim_{n\to\infty} \widetilde{\inf d}\,(x_{\lambda_n}^n, Fz_\rho)}$$

This is contradiction to (2.8.2), so$\boldsymbol{Fz_\rho = z_\rho}$

2.8. Theorem

Let M be a weakly compact convex subset of a soft Metric space $\tilde{X}$ with the Opial property, F be a self-mapping on M satisfying the condition Bγ ,μ, and the sequence $\{x_{\lambda_n}^n\}$ in M be as defined in Proposition. Then $\{x_{\lambda_n}^n\}$converges weakly to a fixed point of F.

Proof: It is clear that $\tilde{d}(\mathrm{F}x_{\lambda_n}^n, x_{\lambda_n}^n) \to 0$as n →∞.

Since M is weakly compact, there exists a subsequence $\{\boldsymbol{x}^{n_j}_{\lambda_{n_j}}\}$ of $\{\boldsymbol{x}^{n}_{\lambda_n}\}$ and $\boldsymbol{z_\rho} \in \boldsymbol{M}$ such that $\{\boldsymbol{x}^{n_j}_{\lambda_{n_j}}\}$converges weakly to $\boldsymbol{z_\rho}$clearly $\boldsymbol{z_\rho}$ is a fixed point of F.

We assume that $\{\boldsymbol{x}^{n}_{\lambda_n}\}$does not converge weakly to $\boldsymbol{z_\rho}$. Then there is a subsequence $\{\boldsymbol{x}^{n_j}_{\lambda_{n_j}}\}$ of $\{\boldsymbol{x}^{n}_{\lambda_n}\}$ and $\boldsymbol{u_l} \in \boldsymbol{M}$ such that $\{\boldsymbol{x}^{n_j}_{\lambda_{n_j}}\}$ converges weakly to$\boldsymbol{u_l}$ and$\boldsymbol{u_l} \neq \boldsymbol{z_\rho}$. Again, $\boldsymbol{Fu_l = u_l}$

Now $$\lim_{n\to\infty} \widetilde{\inf \mathbf{d}}\,(\mathbf{x}^{\mathbf{n}}_{\lambda_\mathbf{n}}, \mathbf{z}_\rho) = \lim_{\mathbf{n_j}\to\infty} \widetilde{\inf \mathbf{d}}\,(\mathbf{x}^{\mathbf{n_j}}_{\lambda_{\mathbf{n_j}}}, \mathbf{z}_\rho) < \lim_{\mathbf{n_j}\to\infty} \widetilde{\inf \mathbf{d}}\,(\mathbf{x}^{\mathbf{n_j}}_{\lambda_{\mathbf{n_j}}}, \mathbf{u_l})$$

(by opial property)

$$= \lim_{\mathbf{n_j}\to\infty} \widetilde{\inf \mathbf{d}}\,(\mathbf{x}^{\mathbf{n_j}}_{\lambda_{\mathbf{n_j}}}, \mathbf{u_l}) < \lim_{\mathbf{n_j}\to\infty} \widetilde{\inf \mathbf{d}}\,(\mathbf{x}^{\mathbf{n_j}}_{\lambda_{\mathbf{n_j}}}, \mathbf{z}_\rho) = \lim_{\mathbf{n}\to\infty} \inf \mathbf{d}\,(\mathbf{x}^{\mathbf{n}}_{\lambda_\mathbf{n}}, \mathbf{z}_\rho)$$

This is contradiction, so $\{x^n_{\lambda_n}\}$converges weakly toz_ρ.

2.9. Lemma

Let M be a non empty closed and convex subset of a soft Metric space$\widetilde{\boldsymbol{X}}$. Let F be a self Mapping on M satisfying the condition Bγ,μ. For $\boldsymbol{x^0_{\lambda_0}}$ in M, let $\{\boldsymbol{x}^{n}_{\lambda_n}\}$be a sequence in M defined by the above iteration. Then $\boldsymbol{\lim_{n\to\infty}\tilde{d}}(\boldsymbol{x}^{n}_{\lambda_n}, \boldsymbol{z_\rho})$ exist for all $\boldsymbol{z_\rho}$ in F(F).

$$\begin{aligned}
\tilde{d}(x^{n+1}_{\lambda_{n+1}}, z_\rho) &= \tilde{d}(\beta_n F y^n_{\eta_n} + (1-\beta_n) y^n_{\eta_n}, z_\rho) \le \beta_n \tilde{d}(F y^n_{\eta_n}, z_\rho) + \tilde{d}((1-\beta_n) y^n_{\eta_n}, z_\rho) \\
&\le \beta_n \tilde{d}(y^n_{\eta_n}, z_\rho) + (1-\beta_n)\tilde{d}(y^n_{\eta_n}, z_\rho) = \tilde{d}(y^n_{\eta_n}, z_\rho) \\
&= \tilde{d}\left(\alpha_n F x^n_{\lambda_n} + (1-\alpha_n) x^n_{\lambda_n}, z_\rho\right) = \alpha_n \tilde{d}(x^n_{\lambda_n}, z_\rho) + (1-\alpha_n)\tilde{d}(x^n_{\lambda_n}, z_\rho) \\
&= \tilde{d}(x^n_{\lambda_n}, z_\rho)
\end{aligned}$$

That is $\boldsymbol{\tilde{d}(x^n_{\lambda_n}, z_\rho)}$ is non-increasing and bounded sequence. Thus there exist for all $\boldsymbol{z_\rho}$ in F(F).

2.10. Lemma

Let $\tilde{X}$ be a uniformly convex soft Metric space. Let $\{\delta_n\}$ be a sequence of real numbers such that$0 < a \le \delta_n \le \mathrm{b} < 1$, ∀n ∈ N and let $\{x^n_{\lambda_n}\}$ and $\{y^n_{\eta_n}\}$ be sequences in $\tilde{X}$ such that$limsup\, x^n_{\lambda_n} \le r, limsup\, y^n_{\eta_n} \le r$and $lim[\delta_n x^n_{\lambda_n} + (1-\delta_n) y^n_{\eta_n}] = r$ for some $r \ge 0$then $\lim_{n\to\infty} \tilde{d}(x^n_{\lambda_n}, y^n_{\eta_n})=0$

2.11. Theorem

Let M be a nonempty closed convex subset of a uniformly convex soft Metric space$\tilde{X}$. Let F be a self-mapping on M satisfying B γ, μ condition. Let $\{x^n_{\lambda_n}\}$ be a sequence in M defined by the iteration scheme where $\alpha_n, \beta_n \in (0, 1)$. Then F(F) ≠φ if and only if $\{x^n_{\lambda_n}\}$ is bounded and $\lim_{n\to\infty} \tilde{d}(F x^n_{\lambda_n}, x^n_{\lambda_n}) = 0$.

Proof: Let F(F) ≠φ and $\boldsymbol{z_\rho} \in$ F(F)

By lemma 3.15 $\lim_{n\to\infty} \tilde{d}(x_{\lambda_n}^n, z_\rho)$ exist and $\{x_{\lambda_n}^n\}$is bounded.

$$\lim_{n\to\infty} \tilde{d}(x_{\lambda_n}^n, z_\rho) = q \text{ (say)} \qquad (2.12..1)$$

Now

$$\tilde{d}(Fy_{\eta_n}^n, z_\rho) \le \tilde{d}(y_{\eta_n}^n, z_\rho) \text{ (by lemma 3.3)}$$
$$\le \tilde{d}(x_{\lambda_n}^n, z_\rho)$$

So $$\lim_{n\to\infty} \widetilde{\sup d}\,(Fy_{\eta_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(y_{\eta_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, z_\rho) = q \qquad (2.12.2)$$

Also

$$\lim_{n\to\infty} \tilde{d}(x_{\lambda_{n+1}}^{n+1}, z_\rho) = \lim_{n\to\infty} \tilde{d}\left(\beta_n(Fy_{\eta_n}^n + (1-\beta_n)),\ z_\rho\right)$$
$$\le \lim_{n\to\infty} \beta_n \tilde{d}\left(Fy_{\eta_n}^n, z_\rho\right) + \lim_{n\to\infty}(1-\beta_n)\tilde{d}\left(y_{\eta_n}^n, z_\rho\right)\left(\beta_n, z_\rho\right)$$
$$= \lim_{n\to\infty} \tilde{d}(x_{\lambda_n}^n, z_\rho) = q$$

So $\lim_{n\to\infty} \tilde{d}(Fy_{\eta_n}^n, y_{\eta_n}^n) = 0$

Again we have

$$\lim_{n\to\infty} \widetilde{\sup d}\,(Fx_{\lambda_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, z_\rho) = q \qquad (2.12.3)$$

Now

$$\tilde{d}(x_{\lambda_{n+1}}^{n+1}, z_\rho) \le \beta_n\, \tilde{d}(Fy_{\eta_n}^n, z_\rho) + (1-\beta_n)\tilde{d}(y_{\eta_n}^n, z_\rho) \le \beta_n\, \tilde{d}(y_{\eta_n}^n, z_\rho) + (1-\beta_n)\tilde{d}(y_{\eta_n}^n, z_\rho)$$

$\Rightarrow$ $q \le \lim_{n\to\infty} \widetilde{\inf d}\,(y_{\eta_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(y_{\eta_n}^n, z_\rho) \le q$ (using 3.7)

$\Rightarrow$ $\lim_{n\to\infty} \tilde{d}(y_{\eta_n}^n, z_\rho) = q$

So $$\lim_{n\to\infty}\left[\alpha_n\, \tilde{d}(Fx_{\lambda_n}^n, z_\rho) + (1-\alpha_n)\tilde{d}(x_{\lambda_n}^n, z_\rho)\right] = \lim_{n\to\infty} \tilde{d}(y_{\eta_n}^n, z_\rho) = q \qquad (2.12.4)$$

So

$$\lim_{n\to\infty}\left[\tilde{d}(Fx_{\lambda_n}^n, z_\rho) - \tilde{d}(x_{\lambda_n}^n, z_\rho)\right] = 0 \Rightarrow \lim_{n\to\infty} \tilde{d}(Fx_{\lambda_n}^n, x_{\lambda_n}^n) = 0$$

Conversely, let $\{x_{\lambda_n}^n\}$be bounded and $\lim_{n\to\infty} \tilde{d}(Fx_{\lambda_n}^n, x_{\lambda_n}^n) = 0$. Let $z_\rho \in A(M, \{x_{\lambda_n}^n\})$.

$$\tilde{d}(x_{\lambda_n}^n, Fz_\rho) \le (3-m)\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) + \left(1-\frac{m}{2}\right)d(x_{\lambda_n}^n, z_\rho)$$
$$+\mu\left[2\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) + \tilde{d}(x_{\lambda_n}^n, Fz_\rho) + \tilde{d}(z_\rho, Fx_{\lambda_n}^n) + 2\tilde{d}(Fx_{\lambda_n}^n, F^2x_{\lambda_n}^n)\right]$$
$$\le (3-m)\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n) + \left(1-\frac{m}{2}\right)\tilde{d}(x_{\lambda_n}^n, z_\rho)$$
$$+\mu\left[2\tilde{d}(x_{\lambda_n}^n, Fz_\rho) + \tilde{d}(x_{\lambda_n}^n, z_\rho) + \tilde{d}(z_\rho, Fz_\rho) + 2\tilde{d}(x_{\lambda_n}^n, Fx_{\lambda_n}^n)\right]$$

$\Rightarrow$ $(1-\mu)\lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, Fz_\rho) \le \left(1-\frac{m}{2}+\mu\right)\lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, z_\rho)$

$\Rightarrow$ $\lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, Fz_\rho) \le \left(\frac{1-m/2+\mu}{1-\mu}\right)\lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, z_\rho)$

$$\left(\text{as } \frac{1-m/2+\mu}{1-\mu} < 1, \text{ for } 2\mu \le \gamma = \frac{m}{2}\right)$$

$\Rightarrow$ $r(Fz_\rho, \{x_{\lambda_n}^n\} \le r(z_\rho, \{x_{\lambda_n}^n\})$

So, $Fz_\rho \in A(M, \{x_{\lambda_n}^n\})$.

Since $\tilde{X}$ is uniformly convex, so$Fz_\rho = z_\rho$ i.e. $z_\rho \in F(F)$, hence $F(F)$ is not empty

2.12. Lemma

Let F be a self-mapping on a nonempty closed and convex subset M of a Metric space $\tilde{\boldsymbol{X}}$. Let F satisfy the condition Bγ,μ on M. For $\boldsymbol{x_{\lambda_0}^0} \in \boldsymbol{M}$, we define a sequence $\{\boldsymbol{x_{\lambda_n}^n}\}$ in M by the iteration scheme (3.10), where $0 \le \boldsymbol{a_n}, \boldsymbol{b_n} \le \boldsymbol{1}$, then,$\lim_{n\to\infty} \tilde{\boldsymbol{d}}(\boldsymbol{x_{\lambda_n}^n}, \boldsymbol{z_\rho})$ exist for all $\boldsymbol{z_\rho} \in \boldsymbol{F(F)}$.

Proof: Can be proved easily

2.13. Theorem

Let F be a self-mapping on a nonempty closed and convex subset M of a uniformly convex soft metric space $\tilde{\boldsymbol{X}}$. Let F satisfy the condition Bγ ,μ on M. Let $\{\boldsymbol{x_{\lambda_n}^n}\}$be a sequence in M defined by the iteration scheme where $0 \le a_n, b_n \le 1$and$\lim_{n\to\infty} a_n = k(\neq 0)$. Then F(F) ≠φ if and only if $\{\boldsymbol{x_{\lambda_n}^n}\}$ is bounded and $\lim_{n\to\infty} \tilde{d}(Fx_{\lambda_n}^n, x_{\lambda_n}^n) = 0$.

Proof: Let F(F) ≠ φ and $\boldsymbol{z_\rho} \in \boldsymbol{F(F)}$. Then, $\lim_{n\to\infty} \tilde{d}(x_{\lambda_n}^n, z_\rho)$ exists and $\{\boldsymbol{x_{\lambda_n}^n}\}$is bounded

$$\lim_{n\to\infty} \tilde{d}(x_{\lambda_n}^n, z_\rho) = q \text{ (say)} \tag{2.14.1}$$

we have $\tilde{d}(Fy_{\eta_n}^n, z_\rho) \le \tilde{d}(y_{\eta_n}^n, z_\rho) \le \tilde{d}(x_{\lambda_n}^n, z_\rho)$

So $\lim_{n\to\infty} \widetilde{\sup d}\,(Fy_{\eta_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(y_{\eta_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(x_{\lambda_n}^n, z_\rho) = q$ (2.14.2)

Again,

$$\tilde{d}(x_{\lambda_{n+1}}^{n+1}, z_\rho) = \tilde{d}(Fz_{\rho_n}^n, z_\rho) \le \tilde{d}(z_{\rho_n}^n, z_\rho) = \tilde{d}\left((1-a_n)x_{\lambda_n}^n + a_n Fy_{\eta_n}^n,\ z_\rho\right)$$
$$\le a_n\tilde{d}\left(Fy_{\eta_n}^n,\ z_\rho\right) + (1-a_n)\tilde{d}\left(x_{\lambda_n}^n, z_\rho\right)$$
$$\le a_n\tilde{d}\left(y_{\eta_n}^n,\ z_\rho\right) + (1-a_n)\tilde{d}\left(x_{\lambda_n}^n, z_\rho\right)$$
$$\le a_n\tilde{d}\big((1-b_n)x_{\lambda_n}^n + b_nFx_{\lambda_n}^n,\ z_\rho\big) + (1-a_n)\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big)$$
$$\le a_n\big[b_n\tilde{d}\big(Fx_{\lambda_n}^n,\ z_\rho\big) + (1-b_n)\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big)\big] + (1-a_n)\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big)$$
$$\le a_nb_n\tilde{d}\big(Fx_{\lambda_n}^n,\ z_\rho\big) + a_n\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big) - b_n\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big) + \tilde{d}\big(x_{\lambda_n}^n, z_\rho\big) - a_n\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big)$$
$$= \tilde{d}\big(x_{\lambda_n}^n, z_\rho\big)$$
$$= \tilde{d}\big(x_{\lambda_n}^n, z_\rho\big) \tag{2.14.3}$$

Thus
$$\tilde{d}(x_{\lambda_{n+1}}^{n+1}, z_\rho) = \tilde{d}\big(x_{\lambda_n}^n, z_\rho\big) \tag{2.14.4}$$

Again,

$\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big) \le \tilde{d}\big(x_{\lambda_n}^n, z_\rho\big)$, for all n ∈ $N \cup \{0\}$

$$\Rightarrow \lim_{n\to\infty} \widetilde{\sup d}\,(Ex_{\lambda_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,\big(x_{\lambda_n}^n, z_\rho\big) = q \tag{2.14.5}$$

Now we have

$$\tilde{d}(x_{\lambda_{n+1}}^{n+1}, z_\rho) \le a_n\tilde{d}(y_{\eta_n}^n, z_\rho) + (1-a_n)\tilde{d}(x_{\lambda_n}^n, z_\rho)$$
$$\Rightarrow \tilde{d}(x_{\lambda_{n+1}}^{n+1}, z_\rho) - (1-a_n)\tilde{d}(x_{\lambda_n}^n, z_\rho) \le a_n\tilde{d}(y_{\eta_n}^n, z_\rho)$$
$$\Rightarrow q - (1-k)q \le \lim_{n\to\infty} \widetilde{\inf d}\,(y_{\eta_n}^n, z_\rho)$$
$$\Rightarrow q \le \lim_{n\to\infty} \widetilde{\inf d}\,(y_{\eta_n}^n, z_\rho) \le \lim_{n\to\infty} \widetilde{\sup d}\,(y_{\eta_n}^n, z_\rho) \le q$$
$$\Rightarrow \lim_{n\to\infty} \tilde{d}(y_{\eta_n}^n, z_\rho) = q$$

Thus
$$\lim_{n\to\infty}\big[b_n\tilde{d}(Fx_{\lambda_n}^n, z_\rho) + (1-b_n)\tilde{d}\big(x_{\lambda_n}^n, z_\rho\big)\big] = \lim_{n\to\infty} \tilde{d}(y_{\eta_n}^n, z_\rho) = q$$

we have $\lim_{n\to\infty}\tilde{d}(Fx^n_{\lambda_n}, x^n_{\lambda_n}) = 0$

For the converse part, let $\{x^n_{\lambda_n}\}$be bounded and $\lim_{n\to\infty}\tilde{d}(Fx^n_{\lambda_n}, x^n_{\lambda_n}) = 0$

Let $z_\rho \in A(M, \{x^n_{\lambda_n}\})$.

$$\lim_{n\to\infty}\widetilde{\sup d}\,(x^n_{\lambda_n}, Fz_\rho) \le \left(\frac{1-\frac{m}{2}+\mu}{1-\mu}\right)\lim_{n\to\infty}\widetilde{\sup d}\,(x^n_{\lambda_n}, z_\rho) \le \lim_{n\to\infty}\widetilde{\sup d}\,(x^n_{\lambda_n}, z_\rho)$$

$\left(as\ \frac{1-m/2+\mu}{1-\mu} < 1\text{, for } 2\mu \le \gamma = \frac{m}{2}\right)$

⇨ $r(Fz_\rho, \{x^n_{\lambda_n}\} \le r(z_\rho, \{x^n_{\lambda_n}\})$

Hence$Fz_\rho \in A(m, \{x^n_{\lambda_n}\})$, $\tilde{X}$being uniformly convex, $Fz_\rho = z_\rho$, i.e. $z_\rho \in F(F)$ Hence F(F) ≠ φ.

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