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Topological Structure of Complex

Pythagorean Fuzzy Sets

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Abstract

The novel concept of topological spaces in complex Pythagorean fuzzy environment is addressed in this paper. Also we introduced the notions of interior and closure operators of a complex Pythagorean fuzzy topological spaces and the properties of these operators are verified as a justification of these operators. The definitions of a complex Pythagorean fuzzy subspace and bases for the topology are given. Also a continuous mappings on complex Pythagorean fuzzy sets are deliberated. A multiple attribute decision making (MADM) algorithm using complex Pythagorean fuzzy topological spaces is defined and investigated with a numerical example as an application for the provided topological spaces.

Keywords. Pythagorean fuzzy sets, complex Pythagorean fuzzy sets, complex Pythagorean fuzzy topology, interior and closure, continuous functions.
MSC: 03E72, 94D05, 54C05.

Introduction

Zadeh [1] developed the theory that deals uncertainty of an object known as fuzzy set (FS).

This theory has got very many applications and extensions. One among them is Atanassov's

[2] intuitionistic fuzzy set (IFS) theory by including two components, namely membership and non-membership grades. Many scientists manifested new speculations via IFS. But, IFS somewhat restricts the researchers in a fixed scope, i.e., one cannot have the membership (m) as 0.7 and non-membership (n) grades as 0.6 simultaneously, as their sum (m+n) outdo the unity. Understanding this, Yager [3] introduced the concept of Pythagorean fuzzy set (PyF set) theory, which also contains the membership (m) and non-membership (n) grades but with an upgraded constraint to meet the more general condition, i.e., $m^2+n^2 \leq 1$, which is a new class of unconventional type of fuzzy sets and has got numerous potential implementations in social and natural sciences (see, e.g. [4-20]). However, FSs, IFSs and PyFs can handle the uncertainty and vagueness that exist in the data. But, the partial ignorance and fluctuations existing in the data, such as phase change or periodicity, were not discussed in FSs, IFSs, PyFSs. To overthrow this, the idea as an extension of FS theory, the notion of a complex fuzzy set (CFS) was put forward by Ramot et al. [21] by extending the range of membership functions with the unit disc from real to complex number. Later, the CFS concept has been generalised to complex intuitionist fuzzy sets (CIFSs) defined by Alkouri and Salleh [22,23] by putting forward the degree of complex valued non-member functions. Recently, Ullah et al. [24] introduced the notion of complex Pythagorean fuzzy set (CPyF set), which extended the design of PFS from actual membership and non membership grades to complex membership and complex non-membership grades, which reduced the information loss. This triggered the researchers to concentrate more on CPyF sets. Akram et al. [25] defined and studied a novel decision-making approach and E. Ma et al., [26] introduced and investigated group decision-making (GDM) approach using CPyF Environment.

Pythagorean fuzzy open sets in Pythagorean fuzzy topological space is defined by Murat Olgun [27]. Similarity measure is way to study the relationship between two object. Multiple attribute decision making (MADM) problem is a vital role in engineering, management, medicine and science. Researchers [28-31] initiated the concept of similarity measure for Pythagorean fuzzy set and an algorithm is proposed to make optimum decision. Alkouri [22] introduced the concept of Complex intuitionistic fuzzy set (CIFS).
Furthermore, a complex Pythagorean fuzzy set (CPyF set) is a more in-discriminated category of the existing theoretical concepts such as FSs, IFSs, CFSs and CIFSs and also driven by the work of [24, 27], a new complex Pythagorean topological spaces (CPyFTS) are defined and their properties are investigated also, the notions of a complex Pythagorean fuzzy bases, subspace, for the CPyFT and also studied continuous mappings on complex Pythagorean fuzzy sets. This blockbuster concept is used in multiple attribute decision making (MADM).

This research article consists of six sections. Introduction is given in section 1. Some prerequisites related to PyF set and CPyF set along with their basic set operations are described in section 2. Definitions of CPyFTS, subspace, base, subbase, neighborhood of a CPyF set, interior and closure are furnished and also investigated their properties in section 3. Continuous functions of CPyF set is deliberated with properties and also a new score function is defined to solve a MADM problem in section 4. Algorithm for MADM problem using CPyFTS is presented in section 5. In section 6, the application of our proposed algorithm is described clearly and a numerical example and in section 7, we finally draw some conclusions and elaborate on future directions.

Preliminaries

The definitions from [3,24] are used in sequel.

Definition 2.1. [3] A PyFS T is defined as T = $\{(m_T(\omega), n_T(\omega)) : \omega \in U\}$, where m_T and n_T represents the membership and non-membership grades, respectively, with a constraint that $0 \leq \mathfrak{m}_{\mathfrak{J}}^2 + \mathfrak{n}_{\mathfrak{J}}^2 \leq 1$. Moreover, the term $r_{\mathfrak{J}}^2 = 1 - (\mathfrak{m}_{\mathfrak{J}}^2 + \mathfrak{n}_{\mathfrak{J}}^2)$ is referred as hesitancy degree. Simply, the pair $(m_{\mathfrak{J}}^2, \mathfrak{n}_{\mathfrak{J}}^2)$ is considered as Pythagorean fuzzy number (PyFN).

Definition 2.2. [24] A complex Pythagorean fuzzy set (CPyF set) P defined over the universe of discourse C is an object of the form

$$P = \{(\omega, h\mu P(\omega)ej\alpha P(\omega), \nu P(\omega)ej\beta P(\omega)i) : \omega \in C\},$$

where j = $\sqrt{-1}$, $\mu_P(\omega), \nu_P(\omega) \in [0,1]$, such that $0 \leq \mu^2{}_P(\omega)+\nu_P{}^2(\omega) \leq 1$ and $\alpha_P(\omega), \beta_P(\omega) \in$ $[0,2\pi]$, such that $0 \leq \alpha_{\mathfrak{P}^2}(\omega) + \beta_{\mathfrak{P}^2}(\omega) \leq 2$.

The values $\mu_P(\omega), \nu_P(\omega)$ and $\alpha_P(\omega), \beta_P(\omega)$ are respectively the amplitude terms and the phase terms.
Simply, the pair $(\mu e^{j\alpha}, \nu e^{j\beta})$ is said to be the complex Pythagorean fuzzy number (CPyFN), where $\mu, \nu \in [0,1]$ and $\alpha, \beta \in [0,2\pi]$, such that $0 \leq \mu^2 + \nu^2 \leq 1$ and $0 \leq \alpha^2 + \beta^2 \leq 2\pi$.

Definition 2.3. [24] Two elements $P_1 = \{(\omega, h\mu_{P1}(\omega)e^{j\alpha P1(\omega)}, \nu_{P1}(\omega)e^{j\beta P1(\omega)}i) : \omega \in C\}$ and $P_2 = \{(\omega, h\mu_{P2}(\omega)e^{j\alpha P2(\omega)}, \nu_{P2}(\omega)e^{j\beta P2(\omega)}i) : \omega \in C\}$ are the two complex Pythagorean sets defined on C, the universe of discourse, then

1. The union of P1 and P2 is P1 ∪P2 = $\{(\omega, h(\mu P1(\omega) \vee \mu P2(\omega))ej(\alpha P1(\omega) \vee \alpha P2(\omega)), (\nu P1(\omega) \wedge$ $\nu_P 2(\omega))ej(\beta P_1(\omega) \wedge \beta P_2(\omega)) : \omega \in C\}$.

2. The intersection of P_1 and P_2 is $P_1 \cap \mathfrak{P}_2 = \{(\omega, \langle(\mu_{\mathfrak{P}_1}(\omega) \wedge \mu_{\mathfrak{P}_2}(\omega))e^{j(\alpha_{\mathfrak{P}_1}(\omega) \wedge \alpha_{\mathfrak{P}_2}(\omega))}, (\nu_{\mathfrak{P}_1}(\omega) \vee$ $\nu_P 2(\omega))ej(\beta P_1(\omega) \vee \beta P_2(\omega)) : \omega \in C\}$.

3. The set P_1 subset P_2 is $P_1 \subseteq P_2$ if and only if $\mu_{P1}(\omega) \leq \mu_{P2}$ and $\nu_{P1}(\omega) \geq \nu_{P2}(\omega)$ for amplitude terms and $\alpha_{P1}(\omega) \leq \alpha_{P2}(\omega)$ and $\beta_{P1}(\omega) \geq \beta_{P2}(\omega)$ for phase terms, for all $\omega \in C$.

4. The set P_1 equal P_2 is $P_1 = P_2$ if and only if $\mu_{P1}(\omega) = \mu_{P2}$ and $\nu_{P1}(\omega) = \nu_{P2}(\omega)$ for amplitude terms and $\alpha_{P1}(\omega) = \alpha_{P2}(\omega)$ and $\beta_{P1}(\omega) = \beta_{P2}(\omega)$ for phase terms, for all $\omega \in C$.

5. The complement of the set P_1 is $P_1{}^c = \{(\omega, \langle \nu_{\mathfrak{P}_1}(\omega)e^{j\beta_{\mathfrak{P}_1}(\omega)}, \mu_{\mathfrak{P}_1}(\omega)e^{j\alpha_{\mathfrak{P}_1}(\omega)}\rangle) : \omega \in \mathfrak{C}\}$.

Complex Pythagorean Fuzzy Topological Spaces

Definition 3.1. Let P be a family of CPyF set. Then $(P;T)$ is said to be a CPyFTS if it satisfies the following:

- 0_P and 1_P are member of P.
- Arbitrary union of CPyF set P in P if each P in P
- Finite intersection of CPyF set P in P if each P in P, where

$1_{\mathfrak{P}} = (1.e^{j.0}, 0.e^{j.2\pi})$ and $0_{\mathfrak{P}} = (0.e^{j.2\pi}, 1.e^{j.0})$. Then $(P;T)$ is called a complex Pythagorean fuzzy topological space (CPyFTS) on P and each CPyF set $P \in P$ is called a complex Pythagorean fuzzy open set (CPyFOS). The complement P^c of a complex Pythagorean fuzzy open set CPyFOS P in a CPyFTSs $(P;T)$ is called a complex Pythagorean fuzzy closed set (CPyFCS) in P.

Example 3.1. Let P = {d} be the universe of discourse. Let D = hd,$(0.7e^{j\pi0.6}, 0.5e^{j\pi0.8})$i be a CPyF set in P. Then $T = \{0_P, 1_P, D\}$ is a CPyFT and the pair (P,T) is a CPyFTS.
Also, D is a CPyFOS in P and its complement $D^c = \langle \mathfrak{d}, (0.5e^{j\pi0.8}, 0.7e^{j\pi0.6})\rangle$ is a CPyFCS in P.

Definition 3.2. Let $(P;T_1)$, $(P;T_2)$ be two PFTSs on P. Then $T_1 \subseteq T_2$, For every $G \in T_1$ implies $G \in T_2$. In this case, we can also state that T_1 is coarser than T_2.

Theorem 3.1. Let the family of topologies $\{T_i : i \in I\}$ of CPyF sets on P. Then $\cap_i\{T_i : i \in I\}$ is also a CPyFTS on P.
Proof: Straight forward.

Theorem 3.2. Let the family of topologies $\{T_i : i \in I\}$ of CPyF with respect to P. Then, a complete lattice is formed by $\{T_i : i \in I\}$ with respect to set inclusion relation of which T^1 and T^o are the largest and the smallest elements respectively.

Proof: Straight forward.

Definition 3.3. Let (P;*T*) be a CPyFTS on P. A sub-collection B ∈ *T* is called as a base for *T* if every member of *T* can be articulated as a union of members of B.

Definition 3.4. Let (P;*T*) be a CPyFTS on P. A sub-collection *S* of *T* is called as a sub-base for *T* if the family of all finite intersections of members of *S* forms a base for *T* .

Theorem 3.3. Let P be a CPyF set and B ⊂ P such that $0_p, 1_p \in$ B. If for any two subsets B_1, B_2 in B and for some $C_x \in B_1 \cap B_2$, ∃Y ∈ B such that $C_x \in Y \subseteq B_1 \cap B_2$, then
B ⊂ P is a base for some topology of CPyF sets.

Proof: Let *T* = ∪{B}. Then

1. $0_p, 1_p \in T$
2. *T* is closed with respect to any union of subsets,
3. Let D,E ∈ *T* . Then $D = \cup_{i \in I} D_i, E = \cup_{j \in J} E_j$. Now $D \cap E = (\cup_{i \in I} D_i) \cap (\cup_{j \in J} E_j) = \cup_{(i,j) \in I \times J} (D_i \cap E_j)$. Now, for $C_x \in (D_i \cap E_j)$, by the hypothesis, 3 Y ∈ B, 3 $C_x \in Y \subset$
$(D_i \cap E_j)$

Since $D_i \cap E_j = \cap\{C_x : C_x \in D_i \cap E_j\} \Rightarrow D_i \cap E_j = \cup\{B\}$. Hence D ∩ E is represented as a union of members of B and thus D ∩ E ∈ *T* . Therefore, B is a base for *T* , which is a topology of CPyF sets on P .

Definition 3.5. Let D and G be two subsets of a CPyTS (P;*T*). Then, G is defined as a neighbourhood (nbd) of D if there exists an open complex Pythagorean fuzzy subset E such
that D ⊂ E ⊂ G.

Theorem 3.4. A complex Pythagorean fuzzy subset D is a open set in a CPyFTS iff it contains a nbd of its each subset.

Definition 3.6. Let (P;*T*) be any CPyFTS with respect to complex Pythagorean fuzzy subset of P. Let D ⊆ P. The complex Pythagorean fuzzy interior and complex Pythagorean fuzzy closure of D are denoted and defined, respectively as stated below:

1. *CPyF-int*(D)=D^o = ∪{G : G is a CPyFOS in P and G ⊆ D},
2. *CPyF-cl*(D)=D^- = ∩{G : G is a CPyFCS in P and G ⊇ D}.

Remark 3.1. Let D in (P;*T*) be any CPyF set, we have

1. [D*c*]CPyF− = [DCPyFo]*c*.
2. [D*c*]CPyFo = [DCPyF−]*c*.
3. D is a CPyFCS if and only if D^{CPyF-} = D.
4. D is a CPyFOS if and only if D^{CPyFo} = D.

5. D^{CPyF-} is a CPyFCS in P.

6. D^{CPyFo} is a CPyFOS in P.

Theorem 3.5. Let (P;T) be a CPyFTS with respect to P where P is a complex Pythagorean fuzzy subset of P. Let D_1 and D_2 be complex Pythagorean fuzzy subsets of P. Then the following statements hold:

1. D ⊆ DCPyF−.

2. D is complex Pythagorean fuzzy closed if and only if D^{CPyF-} = D.

3. 0CPyFp − = 0p and 1CPyFp − = 1p.

4. $D_1 \subseteq D_2 \Rightarrow D_1^{CPyF-} \subseteq D_2^{CPyF-}$.

5. ($D_1 \cup D_2$)CPyF− = D_1CPyF−∪ D_2CPyF−.

6. ($D_1 \cap D_2$)CPyF− = D_1CPyF−∩ D_2CPyF−.

7. (DCPyF−)CPyF− =DCPyF−.

Proof.

1. By definition of complex Pythagorean fuzzy closure, D ⊆ D^{CPyF-}

2. If D is a CPyFCS, then D is the smallest CPyFCS containing itself and hence D^{CPyF-} = D. Conversely, if D^{CPyF-} = D, then D is the smallest complex Pythagorean fuzzy closed set containing itself and hence D is a complex Pythagorean fuzzy closed set.

3. Since 0_p and 1_p are complex Pythagorean fuzzy closed sets in (P;T), $0^{CPyF}{}_p{}^{-} = 0_p$ and $1CPyF_p - = 1_p$.

4. If CPyF set D_1 is a subset of CPyF set D_2, since CPyF set $D_2 \subset D_2^{CPyF-}$, then CPyF set $D_1 \subset D_2^{CPyF-}$. That is, D_2^{CPyF-} is a CPyFCS containing D_1. But D_1^{CPyF-} is the smallest CPyFCS containing D_1. Therefore, $D_1^{CPyF-} \subseteq D_2^{CPyF-}$

5. Since CPyF set D_1 is a subset of union of two CPyF sets D_1 and D_2 and CPyF set D_2 is a subset of union of two CPyF sets D_1 and D_2, $D_1^{CPyF-} \subseteq (D_1 \cup D_2)^{CPyF-}$. Then closure of CPyF set D_1 is a subset of closure of union of two CPyF sets D_1 and D_2 and closure of CPyF set D_2 is a subset of closure of union of two CPyF sets D_1 and D_2. Therefore, union of closure of CPyF sets D_1^{CPyF-}, D_2^{CPyF-} is a subset of closure of union of $(D_1,D_2)^{CPyF-}$. By the fact that $D_1 \cup D_2 \subseteq D_1^{CPyF-} \cup D_2^{CPyF-}$, and since $(D_1 \cup D_2)^{CPyF-}$ is the smallest complex Pythagorean fuzzy closed set containing $D_1 \cup D_2$, so (D1 ∪D2)CPyF− ⊆ D1CPyF−∪D2CPyF−. Thus, (D1 ∪D2)CPyF− = D1CPyF−∪D2CPyF−.

6. Since D1 ∩ D2 ⊆ D1 and D1 ∩ D2 ⊆ D2, (D1 ∩ D2)CPyF− ⊆ D1CPyF−∩ D2CPyF−.

7. Since DCPyF− is a CPyFCS, then (DCPyF−)CPyF− = DCPyF−.

Theorem 3.6. (P;T) be a CPyFTS. Let D be a complex Pythagorean fuzzy subset of P. Then

1. 1p − DCPyFo = (1p − D)CPyF−.
2. 1p − DCPyF− = (1p − D)CPyFo.

Theorem 3.7. Let (P;T) be a CPyFTS. Let D_1 and D_2 be complex Pythagorean fuzzy subsets of P. Then the following statements hold:

1. D is a complex Pythagorean fuzzy open iff $D^{CPyFo} = D$.
2. 0CPyFop = 0p and 1CPyFop = 1p.
3. $D_1 \subseteq D_2 \Rightarrow$ DCPyFo$_1$ $\subseteq$ DCPyFo$_2$.
4. $(D_1 \cup D_2)$CPyFo = DCPyFo$_1$ $\cup$ DCPyFo$_2$. 5. $(D_1 \cap D_2)$CPyFo = DCPyFo$_1$$\cap$ DCPyFo$_2$.
6. (DCPyFo)CPyFo = DCPyFo.

Proof.

1. D is a CPyFOS if and only if 1_p–D is a CPyFCS, if and only if $(1_p–D)^{CPyF-} = 1_p$–D, if and only if $1_p - (1_p - D)^{CPyF-} = D$ if and only if $D^{CPyFo} = D$.
2. Since 0_p and 1_p are complex Pythagorean fuzzy open sets in (P;T), $0^{CPyFo}{}_p = 0_p$ and 1CPyFo$_p$ = 1_p.
3. If D1 ⊆ D2, since D2 ⊇ DCPyFo2 , then D1 ⊇ D2CPyFo . That is, DCPyFo2 is a complex Pythagorean fuzzy open set containing D_1. But $D^{CPyFo}{}_1$ is the largest CPyFOS contained in D1. Therefore, D1CPyFo ⊆ D2CPyFo
4. Since D1 ⊆ D1 ∪ D2 and D2 ⊆ D1 ∪ D2, D1CPyFo ⊆ (D1 ∪ D2)CPyFo and DCPyFo2 ⊆

   (D1 ∪ D2)CPyFo. Therefore, DCPyFo1 ∪ DCPyFo2 ⊆ (D1 ∪ D2)CPyFo. By the fact that $D_1 \cup D_2 \subseteq D^{CPyFo}{}_1 \cup D^{CPyFo}{}_2$, and since $(D_1 \cup D_2)^{CPyFo}$ is the largest complex Pythagorean fuzzy open set containing $D_1 \cup D_2$, so $(D_1 \cup D_2)^{CPyFo} \subseteq D_1{}^{CPyFo} \cup D^{CPyFo}{}_2$. Thus, $(D_1 \cup D_2)$CPyFo = DCPyFo$_1$ ∪ DCPyFo$_2$.
5. Since D1 ∩ D2 ⊆ D1 and D1 ∩ D2 ⊆ D2, (D1 ∩ D2)CPyFo ⊆ DCPyFo1 ∩ DCPyFo2 .
6. Since D^{CPyFo} is a complex Pythagorean fuzzy open set, then $(D^{CPyFo})^{CPyFo} = D^{CPyFo}$.

Complex Pythagorean Fuzzy Continuous Functions

Definition 4.1. Let F : (P;T) → (Q;S) be a function, where (P;T) and (Q;S) are any two CPyFTSs. Then F is said to be a complex Pythagorean fuzzy continuous function (CPyFCF), if for each CPyFOS D in (Q;S), $F^{-1}(D)$ is a CPyFOS in (P;T).

Theorem 4.1. Let F : (P;T) → (Q;S) be a function, where (P;T) and (Q;S) are any two CPyFTSs. Then the following statements are equivalent

(i) F is a CPyFCF.

(ii) $F^{-1}(D)$ is a CPyFCS in (P;T), for each CPyFCS D in (Q;S).

(iii) $F(D^{CPyF-}) \subseteq (F(D))^{CPyF-}$, for each CPyF set D in (P;T).

(iv) $(F^{-1}(D))^{CPyF-} \subseteq F^{-1}(D^{CPyF-})$, for each CPyF set D in (Q;S).

Proof: (*i*) ⇔ (*ii*): The proof is Obvious.

(ii) ⇒ (*iii*): Let D be any CPyFCS in Q. Then $(F(D))^{CPyF-}$ is a CPyFCS in Q. By

(*ii*), $F^{-1}(F(D))^{CPyF-}$ is a CPyFCS in P. Also, we know that $F(D) \subseteq (F(D))^{CPyF-}$. Then F−1(F(D)) ⊆ F−1((F(D))CPyF−). Thus (F−1(F(D)))CPyF− = (D)CPyF− ⊆ (F−1((F(D))CPyF−))CPyF−.
Therefore $F(D^{CPyF-}) \subseteq (F(D))^{CPyF-}$.

(iii) ⇒ (*iv*) Let D be any CPyFCS in Q. By (*iii*), $F(F^{-1}(D)^{CPyF-}) \subseteq (F(F^{-1}(D)))^{CPyF-}$.

Thus (F−1(D))CPyF− ⊆ F−1(DCPyF−).

(iv) ⇒ (*i*) Let D be any CPyFOS in Q. Then D^{-1} is a CPyFCS in Q. By (*iv*) $(F^{-1}(D^c))^{CPyF-} \subseteq$ F−1((D*c*)CPyF−) = F−1(D*c*).

Also we know that $(F^{-1}(D^c))^{CPyF-} \supseteq F^{-1}(D^c)$. Then $(F^{-1}(D^c))^{CPyF-} = F^{-1}(D^c)$. Thus D^c is a CPyFCS in P. So D is a CPyFOS in P. Hence F is a CPyFCF function.

Theorem 4.2. Let F : (P;T) → (Q;S) be a function, where (P;T) and (Q;S) are any two CPyFTSs. Then F is a complex Pythagorean fuzzy continuous function if and only if $F^{-1}((D)^{CPyFo}) \subseteq (F^{-1}(D))^{CPyFo}$ for each CPyF set D in (Q;S).

Proof: Let us assume that F is a complex Pythagorean fuzzy continuous function. Let D ⊂ Q. This implies that $(D)^{CPyFo}$ is a complex Pythagorean fuzzy open set in Q. But we know that F is a complex Pythagorean fuzzy continuous function, this shows that $(F^{-1}(D))^{CPyFo}$ is a complex Pythagorean fuzzy open set in P.

Also we have (D)CPyFo ⊆ D ⇒ F−1((D)CPyFo) ⊆ F−1(D). Therefore, (F−1((D)CPyFo))CPyFo = F−1((D)CPyFo) ⊆ (F−1(D))CPyFo. Thus, F−1((D)CPyFo) ⊆ (F−1(D))CPyFo.

Conversely, suppose $F^{-1}((D)^{CPyFo}) \subseteq (F^{-1}(D))^{CPyFo}$ for each CPyF set D in (Q;S). If D is a complex Pythagorean fuzzy open set in Q, then by hypothesis, $F^{-1}(D) \subseteq (F^{-1}(D))^{CPyFo}$ but we know that $F^{-1}(D) \supseteq (F^{-1}(D))^{CPyFo}$. Then $F^{-1}(D) = (F^{-1}(D))^{CPyFo}$. Thus $F^{-1}(D)$ is a CPyFOS in D. So F is a CPyFCF.

Definition 4.2. Let F : (P;T) → (Q;S) be a function, where (P;T) and (Q;S) are any two CPyFTSs. Then, F is called as a complex Pythagorean fuzzy continuous, if for any complex Pythagorean fuzzy subset D of P and for any nbd E of F(D),there exists a nbd G of D such that F(D) ⊆ E.

Theorem 4.3. Let F : (P;T) → (Q;S) be a function, where (P;T) and (Q;S) are any two CPyFTSs. Then the following statements are equivalent

(i) F is a CPyFCF.

(ii) Let D be any CPyF subset of P and E be any CPyF nbd of F(D), then there exists a nbd G of D such that for any E ⊂ G, we have F(E) ⊂ E.

(iii) Let D be any CPyF subset of P and E be any CPyF nbd of F(D), then there exists a nbd G of D such that G ⊂ F^{-1}(E)

(iv) Let D be any CPyF subset of P and E be any CPyF nbd of F(D), F^{-1}(E) is a nbd of D.

Proof: (*i*) ⇒ (*ii*) Assume that (i) holds, and let D be a CPyF set of P and let E be a nbd of F(D). This implies that there exist a nbd G of D, 3 F(U) ⊂ E. And also if for any E ⊂ G, we get F(E) ⊂ F(U) ⊂ E ⇒ F(E) ⊂ E.

(ii) ⇒ (*iii*) Let (ii) holds good, and let D be a CPyF set of P and let E be a nbd of F(D), as in (ii) there exist a nbd G of D such that for any E ⊂ G ⇒ F(E) ⊂ E. Now we have that F^{-1}(F(E)) ⊂ F^{-1}(E), as E is any subset of G ⇒ G ⊂ F^{-1}(E).

(iii) ⇒ (*iv*) Let (iii) holds, and let D be a CPyF set of P and let E be a nbd of F(D), there exist a nbd G of D such that F(U) ⊂ E. Since G is a nbd of D, there exists an open CPyF subset R ⊂ G, 3 D ⊂ R ⊂ G. But we have G ⊂ F^{-1}(E), we can easily show that D ⊂ Q ⊂ F^{-1}(E), this shows that F^{-1}(E) is a nbd of D.

(iv) ⇒ (*i*) Let us assume that (iv) holds, and let D be a CPyF set of P and let E be a nbd of F(D). Then, there exist a nbd G of D, 3 F(U) ⊂ E. Since G is a nbd of D, and also we have F^{-1}(E) is a nbd of D, there exist an open CPyF subset Q of P, 3 D ⊂ Q ⊂ F^{-1}(E). This implies that F(Q) ⊂ F(F^{-1}(E)) = E. Since Q is a open nbd of D. ⇒ F is a CPyFCF.

Definition 4.3. The score of CPyFNs $\zeta_i = (\mu_i e^{j\alpha_i}, \nu_i e^{j\beta_i})$, where i = 1,2,3,...n can be

defined as:

$$s(\zeta) = \frac{1}{2n}\sum_{i=1}^{n}\{(\mu_i^2 - \nu_i^2) + \frac{1}{4\pi^2}[\alpha_i^2 - \beta_i^2]\},$$

where s denotes the score function of ζs and s(ζ) ∈ [−2,2].

Definition 4.4. The accuracy of CPyFNs $\zeta_i = (\mu_i e^{j\alpha_i}, \nu_i e^{j\beta_i})$, where i = 1,2,3,...n can be defined as:

$$a(\zeta) = \frac{1}{2n}\sum_{i=1}^{n}\{(\mu_i^2 + \nu_i^2) + \frac{1}{4\pi^2}[\alpha_i^2 + \beta_i^2]\},$$

where a denotes the accuracy function of ζs and a(ζ) ∈ [0,2].

Definition 4.5. For the comparison of any two CPyFNs $\zeta_1 = (\mu_1 e^{j\alpha 1}, \nu_1 e^{j\beta 1})$ and ζ_2 =

(μ2ejα2,ν2ejβ2)

1. If s(ζ_1) > s(ζ_2), then $\zeta_1 \succ \zeta_2$ (i.e., ζ_1 is superior to ζ_2),

2. If s(ζ_1) = s(ζ_2), then

   - If a(ζ_1) > a(ζ_2), then $\zeta_1 \succ \zeta_2$ (i.e., ζ_1 is superior to ζ_2),

- If $a(\zeta_1) = a(\zeta_2)$, then $\zeta_1 \sim \zeta_2$ (i.e., ζ_1 is equivalent to ζ_2).

Multiple Attribute Decision-Making Via Complex

Pythagorean Fuzzy Data

MADM is a process to find a best solution that has the greatest degree of gratification from a set of suitable alternatives designated by multiple attributes, and these types of MADM problems emerge in many real-time circumstances. In this section, we introduce a new complex Pythagorean fuzzy topological method of approach for decision-making problem with complex Pythagorean fuzzy information. The following steps are being proposed to select the appropriate alternatives with respect to the attributes in the decision-making situation.

Table 1

	C_1	C_2	C_3	.	.	C_β
A_1	Δ_{11}	Δ_{12}	Δ_{13}	.	.	$\Delta 1\beta$
A_2	Δ_{21}	Δ_{22}	Δ_{23}	.	.	$\Delta 2\beta$
A_3	Δ_{31}	Δ_{32}	Δ_{33}	.	.	$\Delta 3\beta$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
A_α	$\Delta\alpha 1$	$\Delta\alpha 2$	$\Delta\alpha 3$	.	.	$\Delta_{\alpha\beta}$

Proposed Technique of Algorithm Input:

Step 1: Input:

Consider the MADM problems with α alternatives $A_1,A_2,......,A_\alpha$ and β attributes $C_1,C_2,......,C_\beta$. Frame the CPyF set values with respect to attributes and alternatives in the form of table 1.

Step 2: Form the complex Pythagorean fuzzy topologies T_α for the alternatives:

- $T_\alpha = \{A,B,C\}$, where $A = \{0_P, 1_P, \Delta_{1\beta}, \Delta_{2\beta}, ..., \Delta_{\alpha\beta}\}$, $B = \{\Delta_{1\beta} \cup \Delta_{2\beta}, \Delta_{1\beta} \cup \Delta 2\beta, ..., \Delta\alpha{-}1\beta \cup \Delta\alpha\beta\}$ and $C = \{\Delta 1\beta \cap \Delta 2\beta, \Delta 1\beta \cap \Delta 2\beta, ..., \Delta\alpha{-}1\beta \cap \Delta\alpha\beta\}$

Step 3: Calculation of the score numbers for $s(T_\alpha)$: The score numbers for $s(T_\alpha)$ with respect to each alternative using the Definition 4.3.

Step 4: Final Decision:

Arrange complex Pythagorean fuzzy score values for the alternatives $s_1 \prec s_2 \prec .. \prec s_\alpha$.

Choose the alternative which got the highest score value among the other.

Numerical Example

In this part of the paper, a numerical example is brought into play to exemplify the validity of the proposed multi-attribute decision making approach.

Consider a set of five tennis players T = {t_1, t_2,t_3,t_4,t_5} for the selection in the national level tennis tournament. Let Q = {ζ_1,ζ_2,ζ_3,ζ_4} be the set of qualities to be considered by the board chairman for selection of the player, where ζ_1=physical fitness, ζ_2=previous records, ζ_3=confidence and ζ_4=stamina. Construct a CPyFTS (P;T) which specify the abilities of players given in Table 2.

Step 1: Input (Problem field selection):

Table 2

	ζ_1	ζ_2	ζ_3	ζ_4
t_1	$\left\langle 0.5e^{j\pi 0.6} + 0.4e^{j\pi 0.7}, \right.$	$\left\langle 0.6e^{j\pi 0.5} + 0.4e^{j\pi 0.3}, \right.$	$\left\langle 0.6e^{j\pi 0.7} + 0.4e^{j\pi 0.3}, \right.$	$\left\langle 0.8e^{j\pi 0.5} + 0.3e^{j\pi 0.5}, \right.$
t_2	$\left\langle 0.6e^{j\pi 0.6} + 0.4e^{j\pi 0.8}, \right.$	$\left\langle 0.9e^{j\pi 0.8} + 0.4e^{j\pi 0.3}, \right.$	$\left\langle 0.5e^{j\pi 0.3} + 0.4e^{j\pi 0.5}, \right.$	$\left\langle 0.7e^{j\pi 0.7} + 0.7e^{j\pi 0.8}, \right.$
t_3	$\left\langle 0.8e^{j\pi 0.6} + 0.5e^{j\pi 0.4}, \right.$	$\left\langle 0.7e^{j\pi 0.5} + 0.6e^{j\pi 0.7}, \right.$	$\left\langle 0.3e^{j\pi 0.3} + 0.4e^{j\pi 0.6}, \right.$	$\left\langle 0.5e^{j\pi 0.6} + 0.1e^{j\pi 0.3}, \right.$
t_4	$\left\langle 0.9e^{j\pi 0.8} + 0.4e^{j\pi 0.3}, \right.$	$\left\langle 0.7e^{j\pi 0.2} + 0.2e^{j\pi 0.3}, \right.$	$\left\langle 0.7e^{j\pi 0.8} + 0.2e^{j\pi 0.3}, \right.$	$\left\langle 0.6e^{j\pi 0.6} + 0.2e^{j\pi 0.3}, \right.$
t_5	$\left\langle 0.3e^{j\pi 0.6} + 0.2e^{j\pi 0.3}, \right.$	$\left\langle 0.7e^{j\pi 0.5} + 0.2e^{j\pi 0.2}, \right.$	$\left\langle 0.9e^{j\pi 0.9} + 0.2e^{j\pi 0.2}, \right.$	$\left\langle 0.6e^{j\pi 0.5} + 0.1e^{j\pi 0.2}, \right.$

Step 2: Form the complex Pythagorean fuzzy topologies T_α for each player:

1. T_1 = {A_1,B_1,C_1}, where

   A1 = {h0.5*ejπ*0.6,0.4*ejπ*0.7i,h0.6*ejπ*0.5,0.4*ejπ*0.3i,h0.6*ejπ*0.7,0.4*ejπ*0.3i,h0.8*ejπ*0.5,0.3*ejπ*0.5i}, B1 = {h0.6*ejπ*0.6,0.4*ejπ*0.3i,h0.8*ejπ*0.6,0.3*ejπ*0.5i,h0.8*ejπ*0.5,0.3*ejπ*0.3i,h0.8*ejπ*0.7,0.3*ejπ*0.3i},
   C1 $= \{\langle 0.5e^{j\pi 0.5}, 0.4e^{j\pi 0.7}\rangle, \langle 0.6e^{j\pi 0.5}, 0.4e^{j\pi 0.5}\rangle\}$.

2. T_2 = {A_2,B_2,C_2}, where

   A2 = {h0.6*ejπ*0.6,0.4*ejπ*0.8i,h0.9*ejπ*0.8,0.4*ejπ*0.3i,h0.5*ejπ*0.3,0.4*ejπ*0.5i,h0.7*ejπ*0.7,0.7*ejπ*0.8i},

   B2 = {h0.6*ejπ*0.6,0.4*ejπ*0.5i,h0.7*ejπ*0.7,0.4*ejπ*0.8i,h0.7*ejπ*0.7,0.4*ejπ*0.5i}, C2 = {h0.5*ejπ*0.3,0.4*ejπ*0.8i,h0.6*ejπ*0.6,0.7*ejπ*0.8i,h0.5*ejπ*0.3,0.7*ejπ*0.8i}.

3. T_3 = {A_3,B_3,C_3}, where

   A3 = {h0.8*ejπ*0.6,0.5*ejπ*0.4i,h0.7*ejπ*0.5,0.6*ejπ*0.7i,h0.3*ejπ*0.3,0.4*ejπ*0.6i,h0.5*ejπ*0.6,0.1*ejπ*0.3i},

   B3 = {h0.8*ejπ*0.6,0.4*ejπ*0.4i,h0.8*ejπ*0.6,0.1*ejπ*0.3i,h0.7*ejπ*0.5,0.4*ejπ*0.6i,h0.7*ejπ*0.6,0.1*ejπ*0.3i}, C3 = {h0.3*ejπ*0.3,0.5*ejπ*0.6i,h0.5*ejπ*0.6,0.5*ejπ*0.4i,h0.3*ejπ*0.3,0.6*ejπ*0.7i,h0.5*ejπ*0.5,0.6*ejπ*0.7i}.

4. $T_4 = \{A_4, B_4, C_4\}$, where

   A $_4 = \{\langle 0.9e^{j\pi 0.8}, 0.4e^{j\pi 0.3}\rangle, \langle 0.7e^{j\pi 0.2}, 0.2e^{j\pi 0.3}\rangle, \langle 0.7e^{j\pi 0.8}, 0.2e^{j\pi 0.3}\rangle, \langle 0.6e^{j\pi 0.6}, 0.2e^{j\pi 0.3}\rangle\}$,

   B $_4 = \{\langle 0.9e^{j\pi 0.8}, 0.2e^{j\pi 0.3}\rangle, \langle 0.7e^{j\pi 0.6}, 0.2e^{j\pi 0.3}\rangle\}$,

   C4 = {h0.7*ejπ*0.2,0.4*ejπ*0.3i,h0.6*ejπ*0.6,0.4*ejπ*0.3i,h0.6*ejπ*0.2,0.2*ejπ*0.3i}.

5. $T_5 = \{A_5, B_5, C_5\}$, where

   A5 = {h0.3*ejπ*0.6,0.2*ejπ*0.3i,h0.7*ejπ*0.5,0.2*ejπ*0.2i,h0.9*ejπ*0.9,0.2*ejπ*0.2i,h0.6*ejπ*0.5,0.1*ejπ*0.2i},

   B5 $= \{\langle 0.7e^{j\pi 0.6}, 0.2e^{j\pi 0.2}\rangle, \langle 0.6e^{j\pi 0.6}, 0.1e^{j\pi 0.2}\rangle, \langle 0.7e^{j\pi 0.5}, 0.1e^{j\pi 0.2}\rangle, \langle 0.9e^{j\pi 0.9}, 0.1e^{j\pi 0.2}\rangle\}$,

   C5 = {h0.3*ejπ*0.5,0.2*ejπ*0.3i,h0.6*ejπ*0.5,0.2*ejπ*0.2i}.

Step 3: Find CPyFSFs (s_α)**:** $s(T_1) = 0.144271$, $s(T_2) = 0.050625$, $s(T_3) = 0.070333$, $s(T_4) = 0.2005$, $s(T_5) = 0.030455$.

Step 4: Final Decision:

Arrange the complex Pythagorean fuzzy score values for the alternatives in run-up order. We get the following sequences $s_5 \prec s_2 \prec s_3 \prec s_1 \prec s_4$. Here the highest score value is s_4. Thus the chairman of the board will choose the suitable player as player-4=t_4 for the national level tournament.

Conclusion

CPyFTSs is defined in this paper and studied their properties. Also we introduced and studied the concept of closure, interior and continuity in CPyFTS. An algorithm is developed to investigate the real time problem using this novel concept of CPyFTSs. To study the relationship between other existing topological spaces with this novel notion will be our future work.

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