International Journal of Modern Agriculture, Volume 10, No.2, 2021
ISSN: 2305-7246

Obtaining a Priori Error Estimate An Approximate Solution for a Parabolic Type Problem with a Divergent Principal Part

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Annotation. The article considers a parabolic-type boundary value problem with a divergent principal part, when the boundary condition contains the time derivative of the required function**.** Such nonclassical problems with boundary conditions containing the time derivative of the required function arise in a number of applied problems, for example, when a homogeneous isotropic body is placed in an inductor of an induction furnace and an electromagnetic wave is incident on its surface, or its points, washed off with a well-mixed liquid. Similar linear problems of parabolic type with boundary conditions containing the time derivative of the required function arise in the study of the thermal regime of the bed of a wide high-water river. Such problems have been little studied, therefore, the study and solution of problems of parabolic type, when the boundary condition contains the time derivative of the desired function, is relevant and in demandIn this article, a generalized solution to the problem under consideration is defined in the space $\widetilde{\mathrm{H}^{1,1}}(Q_T)$. The aim of the study is to obtain an a priori estimate of the error of the approximate solution in the $L_2\big(0;T;H^1(\Omega)\big)$ norm. The proposed boundary value problem is considered under certain conditions for the function involved in the equation and the boundary condition, which allow the existence and uniqueness of the generalized solution. For the numerical solution of the problem under consideration, an approximate solution was constructed by the Bubnov-Galerkin method for the considered nonclassical parabolic problem with a divergent principal part, when the boundary condition contains the time derivative of the desired function. Under certain conditions for the boundary of the domain, as well as for the coefficients and functions involved in the problem under consideration, we obtained an a priori estimate of the error in the approximate solution of the Bubnov-Galerkin method.

Keywords. Mixed problems, quasilinear equation, boundary condition, Galerkin method, generalized solution, parabolic type, approximate solution, error estimates, a priori estimates, coordinate system, monotonicity, stability, inequalities, boundary, domain, scalar product, continuity, error.

Introduction. In investigations of some actual technical problems, it becomes necessary to study mixed problems of parabolic type, when the boundary condition contains the time derivative of the desired function. Problems of this type arise, for instance, when a homogeneous isotropic body is placed in an induction furnace and an electromagnetic wave falls on its surface. Some nonlinear problems of parabolic type with a boundary condition containing a time derivative of the desired function have been considered, for example, in papers [1-3]. The approximate solution according to Galerkin's method and obtaining a priori estimates of the approximate solution for parabolic classical quasilinear problems without time derivative in the boundary condition were studied by many scientists: Mihlin S. G., Douglas J. Jr, Dupont T. , Dench J. E., Jr, Jutchell L., et al. [4-9]. Quasilinear problems when the boundary condition contains a time derivative of the desired function using the Galerkin method have been studied in works [10-13].

Problem statement. In this paper we consider a quasi-linear problem of the parabolic type when the boundary condition contains the time derivative of the desired function:

$$\begin{cases} u_t - \frac{d}{dx_i} a_i(x,t,u,\nabla u) + a(x,t,u,\nabla u) = 0 \quad , \\ a_0 u_t + a_i(x,t,u,\nabla u)\cos(\nu, x_i) = g(x,t,u), \quad (x,t) \in S_t \ , \\ u(x,0) = u_0(x) \ , \ x \in \Omega \end{cases} \qquad (1)$$

$where\ \Omega -$ the bounded region in E_m,
$m = \dim(\Omega) -$ domain dimension Ω, $a_0 = const > 0$

Definition. As a generalized solution from the space $\widetilde{H^{1,1}}(Q_T) = \{u \in H^{1,1}(Q_T): a_0 u_t \in L_2(S_T)\}$ of problem (1), we call the function from $\widetilde{H^{1,1}}(Q_T)$ that satisfies identity

$$\int_{Q_T} (u_t \eta + a_i(x,t,u,\nabla u)\eta_{xi} + a(x,t,u,\nabla u)\eta)\, dxdt + \int_{S_T} (a_0 u_t + g(x,t,u)))\eta)\, dxdt = 0 \qquad (2)$$

$$\forall\, \eta \in H^{1,1}$$

Main results: Consider problem (1) under the following conditions

1) for $(x,t) \in \bar{Q}_T$ and arbitrary u, v, p и q the inequalities

$$|a_i(x,t,u,p) - a_i(x,t,u,q)| \le \mu |p-q|,$$
$$|a_i(x,t,u,p) - a_i(x,t,v,p)| \le L_0(|u|^{\propto} + |v|^{\propto})|u-v| \ ,$$
$$|a(x,t,u,p) - a(x,t,u,q)| \le L_1\big(|u|^{\propto}|p-q| + \big(|p|^{\beta} + |q|^{\beta}\big)|p-q|\big), \qquad \mathbf{(3)}$$
$$|a(x,t,u,p) - a(x,t,v,p)| \le L_2\big(|p|^{2\beta} + |u|^{2\alpha} + |v|^{2\alpha})|u-v|\big),$$
$$|g(x,t,u) - g(x,t,v)| \le L_3(|u|^{\gamma} + |v|^{\gamma})|u-v|) \ ,$$

are true,

where

$$\propto \in \begin{cases} [0;\infty), m = 2 \\ \left[0; \frac{2m}{(m-2)l}\right], m \ge 3 \end{cases} , \qquad l > m = \dim(\Omega) \qquad \mathbf{(4)}$$

$$\gamma \in \begin{cases} [0;\infty), m = 2 \\ \left[0; \frac{2(m-1)}{(m-2)(l-1)}\right], m \geq 3 \end{cases} \quad , \qquad \beta \leq \frac{2}{l}$$

$$\mu, L_{0,} L_{1,} L_{2,} L_3 \ - positive\ constants.$$

In the case of bounded generalized solutions on the space $\hat{H}^{1.1}(Q_T)$ assumption (3) must be replaced by

$$|a_i(x,t,u,p) - a_i(x,t,u,q)| \leq \tilde{\mu}(|u|)|p-q|$$

$$|a_i(x,t,u,p) - a_i(x,t,v,p)| \leq \widetilde{L_0}\big(|u| \quad + |v| \quad\big)|u-v|$$

$$|a(x,t,u,p) - a(x,t,u,q)| \leq [\widetilde{L_1}\big(|u|\big) + L_1\big(|p|^{\beta} + |q|^{\beta}\big)]|p-q| \qquad (\tilde{3})$$

$$|a(x,t,u,p) - a(x,t,v,p)| \leq [\widetilde{L_2}(|u|) + |v|) + L_2\big(|p|^{2\beta}\big)]|u-v|$$

$$|g(x,t,u) - g(x,t,v)| \leq \widetilde{L_3}(|u| + |v|)|u-v| \quad ,$$

where $\widetilde{\mu}(\tau)$, $\widetilde{L_1}(\tau)$ are continuous positive functions from $\tau \geq 0$.

2) The boundary S of the domain Ω is such that the inequalities [14-16] are true

$$\|u\|_{L\tilde{q}(\Omega)} \leq \varepsilon\, \|\nabla u\|^2_{L_2(\Omega)} + C_\varepsilon \|u\|^2_{L_2(\Omega)},$$

$$\tilde{q} = \frac{2l}{l-2}\ , l > m; \qquad (5)$$

$$\|u\|_{L\bar{q}(S)} \leq \varepsilon\, \|\nabla u\|^2_{L_2(\Omega)} + C_\varepsilon \|u\|^2_{L_2(\Omega)}\ ,$$

$$\bar{q} < \frac{2(m-1)}{m-2}$$

Let us construct an approximate Galerkin solution. We take a coordinate system from the space H^1. We will seek an approximate solution U (x, t) in the form

$$U(x,t) = \sum_{k=1}^{n} C_k^n\,(t)\varphi_k(x) \epsilon M \qquad (6)$$

where $C_k^n(t)$ are determined from a system of ordinary differential equations

$$(U_t, \varphi_j)_{\hat{L}_2} + (a_i(x,t,U,\nabla U), \varphi_{jx_i})_\Omega + (a(x,t,U,\nabla U), \varphi_j)_\Omega =$$

$$= (g(x,t,U), \varphi_j)_S\ , \quad j = \overline{1,n} \qquad (7)$$

and the initial conditions

$$U(x,O) - U_0 - "little"$$

Let's write problem (7) in the form

$\int_0^t (U_t, v)_{\widetilde{L_2}} dt \; + \int_0^t \{(a_i(x,t,U,\nabla U), v_{jx_i})_\Omega + (a(\mathrm{x,t,U},\nabla U), v)_\Omega\} \, dt = \int_0^t (g(x,t,U), v)_s dt,$
$\forall v \epsilon M;$ (8) $U(.,0) - U_0 \quad - "little"$

Denote ξ=W-U, η= u-W where u,U solutions of the problem (2), (8) respectively, W is an arbitrary element of the set P_n.

Subtract equation (8) from equation (2). For a test function v we take v=ξ(.,t). Setting ζ = η + ξ, we arrive at the identity

$$\int_0^t (\xi_t, \xi)_{\widetilde{L_2}} dt \; + \int_0^t \{(a_i(\mathrm{x,t,U},\nabla W) - a_i(\mathrm{x,t,U},\nabla U), \xi_{x_i})_\Omega \; + (\mathrm{a(x,t,u,\nabla u)} -$$
$$-\mathrm{a}(\mathrm{x,t,U},\nabla U), \xi)_\Omega\} \, dt = \int_0^t \Big\{(a_i(\mathrm{x,t,U},\nabla W) - a_i(\mathrm{x,t,u},\nabla W), \xi_{x_i})_\Omega + ((a_i(\mathrm{x,t,u},\nabla W) -$$
$$(a_i(\mathrm{x,t,u,\nabla u}), \xi_{x_i})_\Omega - (\frac{\partial \eta}{\partial t}, \xi)_{\widetilde{L_2}} + ((g(x,t,u) - -g(x,t,U), \xi)_s\Big\} \, dt \qquad (9)$$

Let's use the estimates:

$$\int_\Omega |v|^{2\alpha} |\xi||\eta| dx \le \varepsilon \|\nabla \xi\|^2_{L_2(\Omega)} + C_0 \left(\|\xi\|^2_{L_2(\Omega)} + \|\eta\|^2_{H^1(\Omega)}\right)$$

$$\int_\Omega |v|^{2\alpha} \xi^2 dx \le \varepsilon \|\nabla \xi\|^2_{L_2(\Omega)} + C_1 \|\xi\|^2_{L_2(\Omega)} \qquad (10)$$

$$\int_S |v|^{\gamma} \xi^2 dx \le \varepsilon \|\nabla \xi\|^2_{L_2(\Omega)} + C_2 \|\xi\|^2_{L_2(\Omega)}$$

where α, γ satisfy relation (4), constants C_0, C_1, C_2 depend on the norm $\|v\|^2_{H^1(\Omega)}$. , γ satisfy relation (4), constants C_ (0,) C_ (1,) C_ (2) depend on the norm 〖‖v‖〗 _(H ^ 1 (Ω)) ^ 2.

Inequality (10) is easily obtained if we take into account the embeddings $H^1(\Omega) \subset L_{l\alpha}(\Omega), L_{(l-1)\gamma}(S)$, inequalities (5) and Hölder [17-21]

Indeed, for example

$$\int_\Omega |v|^{2\alpha} |\xi||\eta| dx \le \|v\|^{2x}_{L_{l\alpha}(\Omega)} \|\xi\|_{L_{\frac{2l}{l-2}}(\Omega)} \|\eta\|_{L_{\frac{2l}{l-2}}(\Omega)} \le C_\varepsilon \|v\|^{4\alpha}_{L_{l\alpha}(\Omega)} \|\eta\|^2_{L_{\frac{2l}{l-2}}(\Omega)} + \varepsilon \|\xi\|^2_{L_{\frac{2l}{l-2}}(\Omega)} \le$$
$$C(\|v\|_{H^1(\Omega)}) \Big[\|\eta\|^2_{H^1(\Omega)} + \|\xi\|^2_{L_2(\Omega)}\Big] + +\varepsilon \|\nabla \xi\|^2_{L_2(\Omega)} \; .$$

Then, by assumption (3), we have

$$\left|(a_i(\mathrm{x,t,U},\nabla W) - a_i(\mathrm{x,t},u,\nabla W), \xi_{x_1})_\Omega\right| \le \varepsilon \|\nabla \xi\|^2_{L_2(\Omega)} + C_\varepsilon \int_\Omega (|u|^{2\alpha} + |U|^{2\alpha})(\xi^2 + \eta^2) dx \le$$
$$2\varepsilon \|\nabla \xi\|^2_{L_2(\Omega)} + C\left(\|\xi\|^2_{L_2(\Omega)} + \|\eta\|^2_{H^1(\Omega)}\right), \qquad (11)$$

where $C = C(\|u\|_{H^1(\Omega)}, \|U\|_{H^1(\Omega)})$.

Similarly,

$$\left|(a_i(\mathrm{x,t},u,\nabla W) - a_i(\mathrm{x,t},u,\nabla u), \xi_{x_i})_\Omega\right| \le \varepsilon \|\nabla \xi\|^2_{L_2(\Omega)} + C_1 \|\nabla \eta\|^2_{L_2(\Omega)}.$$

Further,

$$|(a(x,t,u,\nabla u)-a(x,t,U,\nabla U),\xi)_\Omega|$$

$$\leq L_1\int_\Omega (|u|^\alpha+|\nabla u|^\beta+|\nabla U|^\beta)|\nabla(\eta+\xi)||\xi|dx+L_2\int_\Omega (|u|^{2\alpha}+$$

$$+|U|^{2\alpha}+|\nabla U|^{2\beta})|(\eta+\xi)||\xi|dx\leq$$

$$\leq \varepsilon\int_\Omega (\nabla\xi+\nabla\eta)^2dx+C\int_\Omega (|u|^{2\alpha}+|\nabla u|^{2\beta}+|\nabla U|^{2\beta})\xi^2dx+$$

$$+L_2\int_\Omega (|u|^{2\alpha}+|U|^{2\alpha}+|\nabla U|^{2\beta})|\eta||\xi|dx\leq$$

$$\leq 4\varepsilon\|\nabla\xi\|^2_{L_2(\Omega)}+C_2\left(\|\xi\|^2_{L_2(\Omega)}+\|\eta\|^2_{H^1(\Omega)}\right) \qquad (12)$$

And finally

$$|(g(x,t,u)-g(x,t,U),\xi)_S|\leq 2L_3\int_S (|u|^\gamma+|U|^\gamma)\xi^2dx+\frac{1}{4}L_3\int_S (|u|^\gamma+|U|^\gamma)\eta^2dx\leq$$
$$\varepsilon\|\nabla\xi\|^2_{L_2(\Omega)}+C_3(\|\xi\|^2_{L_2(\Omega)}+\|\eta\|^2_{H^1(\Omega)}) \qquad (13)$$

The variables C_1, C_2, C_3 depend on the quantities $\|u\|_{H^1(\Omega)}$, $\|U\|_{H^1(\Omega)}$.

We substitute the obtained estimates into (9); then taking $\xi=\frac{v}{16}$, we have

$$\|\xi\|^2_{\widetilde{L_2}}+\|\nabla\xi\|^2_{L_2(0;t;L_2(\Omega))}\leq C\left[\|\xi(x;0)\|^2_{\widetilde{L_2}}+\|\xi\|^2_{L_2(0;t;\widetilde{L_2})}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L_2(0;t;\widetilde{L_2})}+\|\eta\|^2_{L_2(0;t;H^1(\Omega)}\right]$$

Setting $\xi=\zeta-\eta$ we have

$$\|u-U\|_{L_\infty(0;T;\widetilde{L_2})}+\|u-U\|_{L_2(0;T;H^1(\Omega))}\leq C\left[\|u_0(x)-U(x;0)\|_{\widetilde{L_2}}+\|u-W\|_{L_2(0;T;H^1(\Omega))}+\right.$$
$$\left.\left\|\frac{\partial u}{\partial t}-\frac{\partial W}{\partial t}\right\|_{L_2(0;T;\widetilde{L_2})}\right] \qquad \mathbf{(14)}$$

CONCLUSION: Let us formulate the result obtained in the form of a theorem.

Theorem. Let u,U – be solutions to problem (2), (8), respectively..

Suppose that conditions (3) are satisfied. Then there exists a constant C depending on the quantities m, v, $\|u\|_{H^1(\Omega)}$, $\|U\|_{H^1(\Omega)}$ and on the constants in (3-4) such that for error $\xi=u-U$ estimate (14) holds, where W is an arbitrary function of the form $(x,t)=\sum_{k=1}^n d_k(t)\varphi_k(x)\in M$.

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